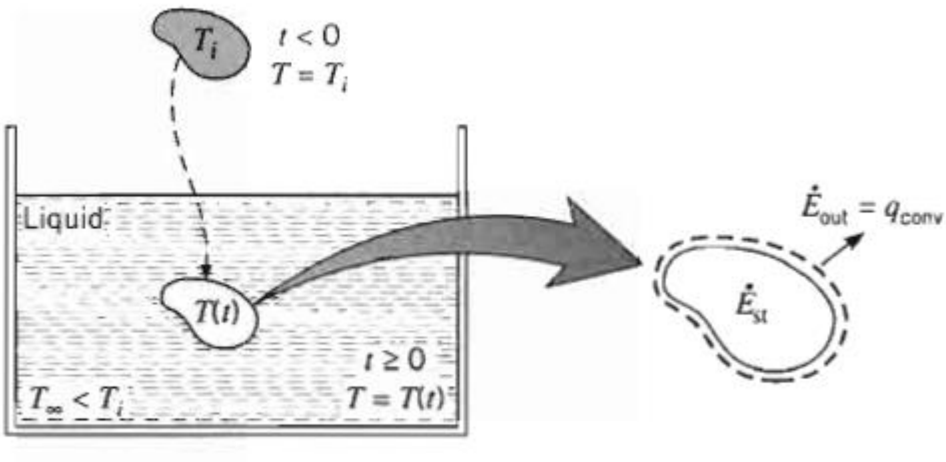
Transient (time dependent) conduction:

The Lumped Capacitance Method: Temperature distribution is spatially uniform



$$-\dot{E}_{out} = \dot{E}_{st}$$

or

$$-hA_s(T - T_\infty) = \rho V c \frac{dT}{dt}$$

Introducing the temperature difference

$$\theta \equiv T - T_\infty$$

and recognizing that $(d\theta/dt) = (dT/dt)$ if T_∞ is constant, it follows that

$$\frac{\rho V c}{hA_s} \frac{d\theta}{dt} = -\theta$$

Separating variables and integrating from the initial condition,
for which $t = 0$ and $T(0) = T_i$, we then obtain

$$\frac{\rho V c}{hA_s} \int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = -\int_0^t dt$$

where

$$\theta_i \equiv T_i - T_\infty$$

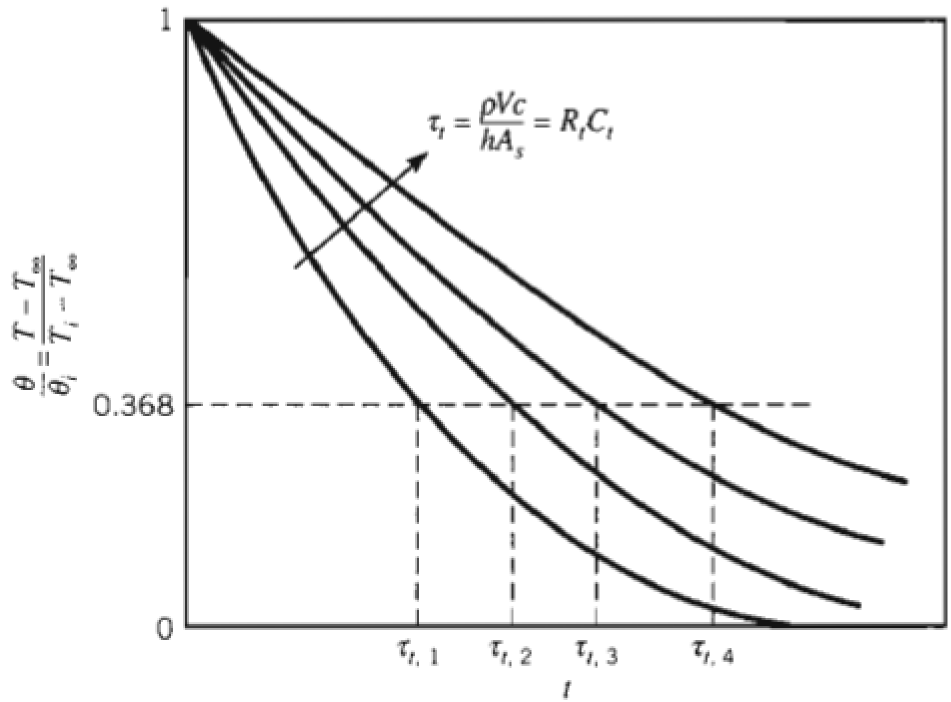
$$\frac{\rho V c}{h A_s} \ln \frac{\theta_i}{\theta} = t$$

or

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{h A_s}{\rho V c}\right) t\right]$$

The quantity $\rho Vc/hA_s$ may be interpreted as *thermal time constant* expressed as

$$\tau_t = \left(\frac{1}{h A_s}\right)(\rho V c) = R_t C_t$$



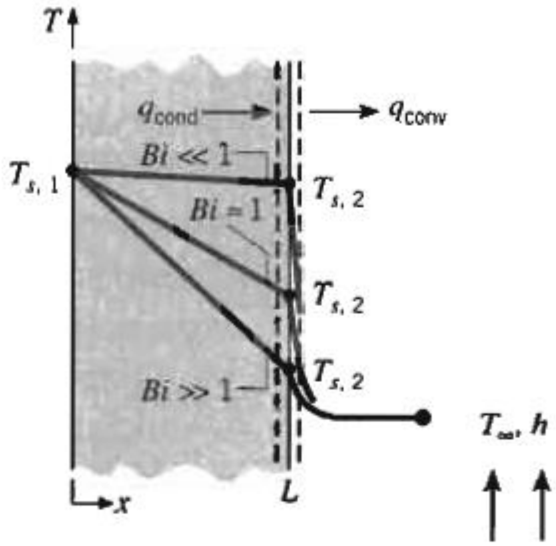
Total energy transfer occurring up to time t

$$Q = \int_0^t q\, dt = h A_s \int_0^t \theta\, dt$$

Substituting for θ

$$Q = (\rho V c)\theta_i \left[1 - \exp\left(-\frac{t}{\tau_t}\right)\right]$$

Validity of the Lumped Capactance Method:

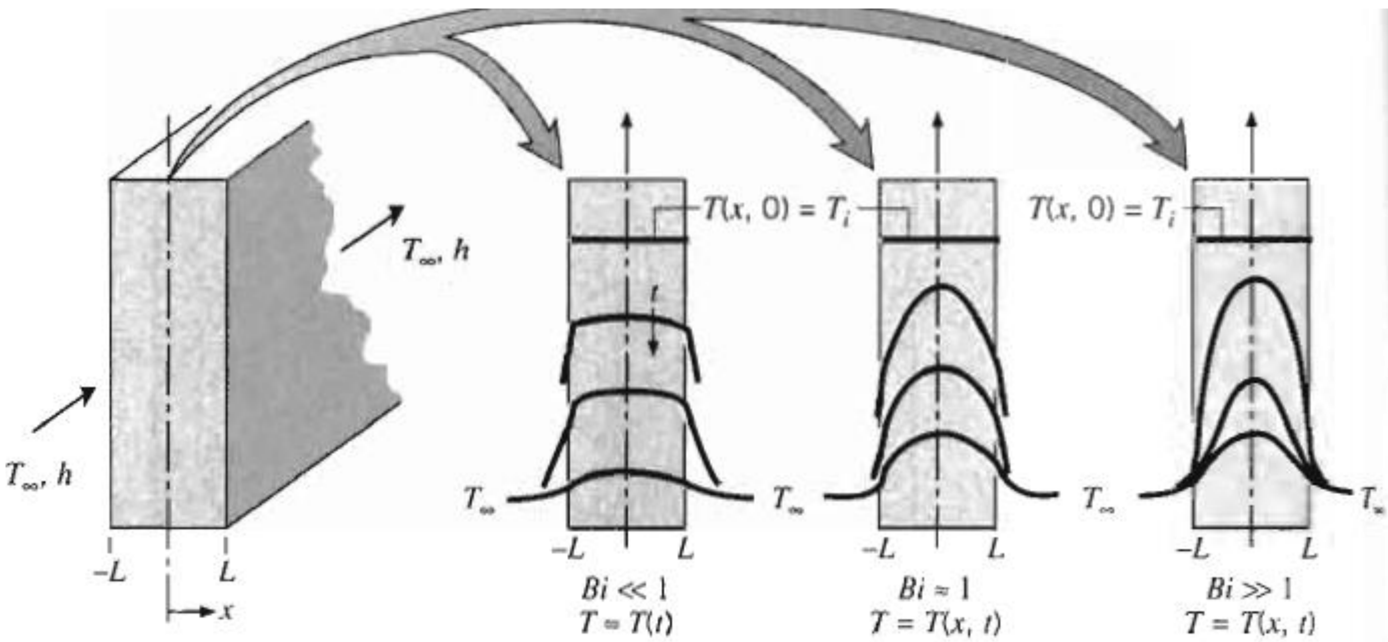


The surface energy balance for this situation;

$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)$$

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{cond}}{R_{conv}} = \frac{hL}{k} \equiv Bi$$

Where *Bi* is the Biot number, defined as the ratio of the conduction resistance to convection resistance.



Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection.

$Bi = \frac{hL_c}{k} < 0.1$ Validity of LCM

$L_c \equiv V/A_s.$

- $L_c = L$ for a plane wall of thickness $2L$
- $L_c = 2r_o$ for long cylinder
- $L_c = 3r_o$ for sphere

Returning to the basic equation, time multiple term $hA_s/\rho Vc$ rearrange as

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho Vc}\right)t\right]$$

With $L_c = V/A_c$

$$\frac{hA_s t}{\rho Vc} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2}$$

or

$$\frac{hA_s t}{\rho Vc} = Bi \cdot Fo$$

where

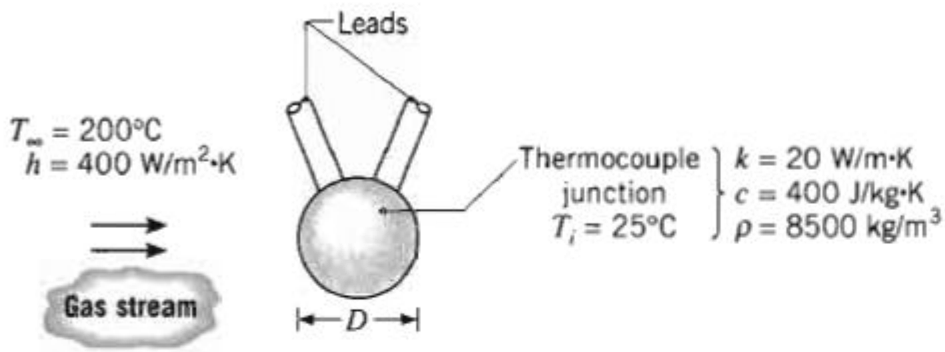
$$Fo \equiv \frac{\alpha t}{L_c^2}$$

Fo is the Fourier number and is referred as dimensionless time. Then,

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo)$$

Example problem:

A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is $h = 400\ \text{W/m}^2 \cdot \text{K}$, and the junction thermophysical properties are $k = 20\ \text{W/m} \cdot \text{K}$, $c = 400\ \text{J/kg} \cdot \text{K}$, and $\rho = 8500\ \text{kg/m}^3$. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?



$$\tau_t = \frac{1}{h\pi D^2} \times \frac{\rho \pi D^3}{6} c$$

$$D = \frac{6h\tau_t}{\rho c} = \frac{6 \times 400\ \text{W/m}^2 \cdot \text{K} \times 1\ \text{s}}{8500\ \text{kg/m}^3 \times 400\ \text{J/kg} \cdot \text{K}} = 7.06 \times 10^{-4}\ \text{m} \quad \triangleleft$$

$$Bi = \frac{h(r_o/3)}{k} = \frac{400\ \text{W/m}^2 \cdot \text{K} \times 3.53 \times 10^{-4}\ \text{m}}{3 \times 20\ \text{W/m} \cdot \text{K}} = 2.35 \times 10^{-3}$$

Time required to reach 199 °C;

$$t = \frac{\rho(\pi D^3/6)c}{h(\pi D^2)} \ln \frac{T_i - T_\infty}{T - T_\infty} = \frac{\rho D c}{6h} \ln \frac{T_i - T_\infty}{T - T_\infty}$$

$$t = \frac{8500\ \text{kg/m}^3 \times 7.06 \times 10^{-4}\ \text{m} \times 400\ \text{J/kg} \cdot \text{K}}{6 \times 400\ \text{W/m}^2 \cdot \text{K}} \ln \frac{25 - 200}{199 - 200}$$

$$t = 5.2\ \text{s} \approx 5\tau_t \quad \triangleleft$$

Example problem:

Stainless steel ball bearings ($\rho = 8085\ kg/m^3$, $k = 15\ W/mK$, $C = 480\ J/kgK$, and $\alpha = 3.865 \times 10^{-6}\ m^2/s$) having a diameter of *1.8 cm* are to be quenched in water. The balls leave the oven at a uniform temperature of *900 °C* and are exposed to air at *30 °C* for a while before they are dropped into the water. If the temperature at the center of the balls is not to fall below *850 °C* prior to quenching and the heat transfer coefficient in the air is *150 W/m²K*, determine how long they can stand in the air before being dropped into the water.

$D = 1{,}8\ cm = 0.018\ m$

$T_\infty = 30°C$

$T_i = 900°C$

$T(0) = 850°C$

$h = 150\ W/m^2K$

$r_o = \frac{D}{2} = 0{,}009$

$$Bi = \frac{h r_o}{3k} = \frac{\overset{10}{\cancel{150}} \times 0{,}009}{3 \cdot \cancel{15}} = \frac{0{,}09}{3} = 0{,}03 < 0{,}1 \quad LCM$$

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot F_o) \ ; \ \ln \frac{T - T_\infty}{T_i - T_\infty} = -Bi\, F_o$$

$$F_o = -\frac{\ln \frac{T - T_\infty}{T_i - T_\infty}}{Bi} = \frac{\ln \frac{850-30}{900-30}}{0{,}03} = \frac{\ln \frac{820}{870}}{0{,}03} = 1.973$$

$$F_o = \frac{\alpha t}{\left(\frac{r_o}{3}\right)^2} = \frac{9 \alpha t}{r_o^2}$$

$$t = \frac{r_o^2 \cdot F_o}{9\alpha} = \frac{(0{,}009)^2 \cdot 1.973}{9 \times 10^{-6} \times 3.865} = \frac{(0{,}009)^2 \times 1.973}{9 \times 3.865} \cdot 10^6$$

$$\boxed{t = 4.59\ s}$$

Spatial effects:

For the case when Bi > 0.1 LCM is not applicable and spatial effects are needed to be taken into account.

Returning to basic equation for Cartesian system

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x, 0) = T_i$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L, t) - T_\infty]$$

$$T = T(x, t, T_i, T_\infty, L, k, \alpha, h)$$

$$\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty}$$

$$x^* \equiv \frac{x}{L}$$

$$t^* \equiv \frac{\alpha t}{L^2} \equiv Fo$$

$$\frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial Fo}$$

$$\theta^*(x^*, 0) = 1$$

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=0} = 0$$

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1} = -Bi\,\theta^*(1, t^*)$$

$$\theta^* = f(x^*, Fo, Bi)$$

Approximate (one-term) solutions:

Case	**Temperature distribution**	**Energy transfer**
Plane wall	$\theta^* = C_1 \exp(-\zeta_1^2 Fo)\cos(\zeta_1 x^*)$	$\frac{Q}{Q_o} = 1 - \frac{\sin\zeta_1}{\zeta_1}\theta_o^*$
Cylinder	$\theta^* = C_1 \exp(-\zeta_1^2 Fo) J_0(\zeta_1 r^*)$	$\frac{Q}{Q_o} = 1 - \frac{2\theta_o^*}{\zeta_1} J_1(\zeta_1)$
Sphere	$\theta^* = C_1 \exp(-\zeta_1^2 Fo)\frac{1}{\zeta_1 r^*}\sin(\zeta_1 r^*)$	$\frac{Q}{Q_o} = 1 - \frac{3\theta_o^*}{\zeta_1^3}[\sin(\zeta_1) - \zeta_1\cos(\zeta_1)]$

Here Q_o is the initial energy of the wall relative to the fluid temperature as flows

$$Q_o = \rho c V(T_i - T_\infty)$$

θ_o^* is to be determine for $x_o = 0$ at centerline temperature distribution as $\theta_o^* = C_1 \exp(-\zeta_1^2 Fo)$.

	Plane Wall		Infinite Cylinder		Sphere	
Bi[a]	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.03	0.1723	1.0049	0.2440	1.0075	0.2991	1.0090
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.05	0.2218	1.0082	0.3143	1.0124	0.3854	1.0149
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.07	0.2615	1.0114	0.3709	1.0173	0.4551	1.0209
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.09	0.2956	1.0145	0.4195	1.0222	0.5150	1.0268
0.10	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.15	0.3779	1.0237	0.5376	1.0365	0.6609	1.0445
0.20	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.25	0.4801	1.0382	0.6856	1.0598	0.8447	1.0737
0.30	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0932	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0919	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5994	1.3384	2.0288	1.4793

[a] $Bi = hL/k$ for the plane wall and hr_o/k for the infinite cylinder and sphere. See Figure 5.6.

Bi^a	Plane Wall ζ_1 (rad)	Plane Wall C_1	Infinite Cylinder ζ_1 (rad)	Infinite Cylinder C_1	Sphere ζ_1 (rad)	Sphere C_1
2.0	1.0769	1.1785	1.5994	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2402	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	1.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2881	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2733	2.4050	1.6018	3.1415	2.0000

[a] $Bi = hL/k$ for the plane wall and hr_o/k for the infinite cylinder and sphere. See Figure 5.6.

Example problem:

A concrete (ρ = 2300 kg/m^3, C_p = 880 J/kg/K and k = 2,5 W/mK) slab of thickness 20 cm having uniform temperature of 55 °C is suddenly subjected to an airstream at 10 °C. Calculate the temperature in the concrete slab at a depth of 7 cm after 1 hour for the cases when;

a) The subjected air stream represents a very mild wind (h = 2 W/m^{2}K)

b) The subjected air stream represents a relatively strong wind (h = 100 W/m^{2}K)

a) $Bi = \frac{hL}{k} = \frac{2\times 0{,}10}{2{,}5} = 0{,}08 < 0.1$ LCM

$T_\infty = 10°C$

$\frac{T-T_\infty}{T_i-T_\infty} = \exp(-Bi \cdot Fo)$

$Fo = \frac{\alpha t}{L^2} = \frac{kt}{\rho c_p L^2}$

$2L = 20$ cm
$L = 0{,}10$ m

$T_i = 55°C$

$t = 3600$ s

$$Fo = \frac{2.5 \times 3600}{2300 \times 860 \times (0.1)^2}$$

$$Fo = 0.444$$

$$\frac{T - T_f}{T_i - T_f} = \exp(-0.08 \times 0.444) = 0.965, \quad T = T_f + (T_i - T_f) \cdot 0.965$$

$$T = 10 + (55 - 10) \cdot 0.965$$

$$\boxed{T = 53.43°C}$$

b) $Bi = \frac{hL}{k} = \frac{100 \times 0.1}{2.5} = 4 > 0.1$

$\varphi_1 = 1.2646$, $C_1 = 1.2287$, $x^* = \frac{x}{L} = \frac{3}{10} = 0.3$

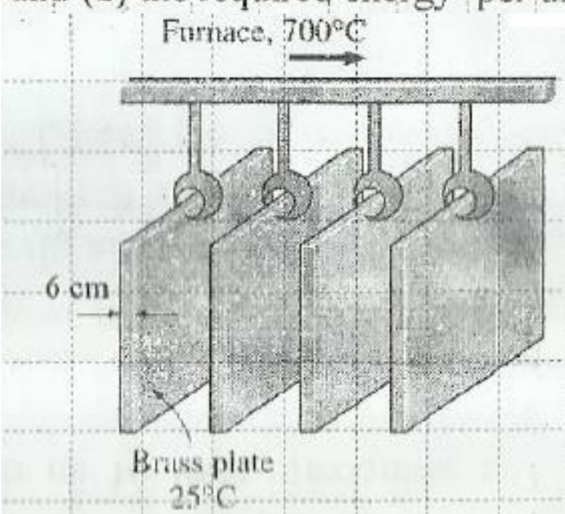
$$\frac{T(x^*) - T_f}{T_i - T_f} = C_1 \cdot \exp(-\varphi_1^2 Fo) \cdot \cos(\varphi_1 x^*)$$

$$T(r^*) = 10 + 45 \times 1.2287 \cdot \exp\left[-(1.2646)^2 \cdot (0.444)\right] \cos(0.3 \times 1.2646)$$

$$\boxed{T(r^*) = 35.45°C}$$

Example problem:

In a production facility, 6-cm-thick large brass plates ($k = 100$ W/mK, $\rho = 8530$ kg/m^3, $C_p = 380$ J/kgK, and $\alpha = 30.85 \times 10^{-6}$ m^2/s) that are initially at a uniform temperature of **25°C** are heated by passing them through an oven maintained at **700°C**. Taking the convection heat transfer coefficient to be **h=1000 W/m²K**, determine; **(a)** the time required to reach the temperature of the layer **1 cm** below the surface to **400°C** and **(b)** the required energy per unit surface area of the plate for this process.



$k = 100 W/mK$
$\rho = 8530 kg/m^3$
$C_p = 380 J/kgK$
$\alpha = 30{,}85 m^2/s$
$T_i = 25°C$
$T_\infty = 700°C$
$h = 1000 W/m^2K$
$T(0.02, t) = 400°C$

$Bi = \frac{hL}{k} = \frac{1000 * 0.03}{100} = 0.3 > 0.1$ One-term solution

$\varphi_1 = 0.5218$, $C_1 = 1.045$ $x^* = \frac{2}{3}$

$$\frac{T(0.02, t) - T_\infty}{T_i - T_\infty} = C_1 \exp(-\varphi_1^2 \cdot Fo) \cdot \cos(\varphi_1 x^*)$$

$$\frac{400-700}{25-700} = 1.045 \cdot \exp[-(0.5218)^2 \cdot Fo] \cdot \cos\left(0.5218 * \frac{2}{3}\right)$$

$$\frac{300}{675} = 0.982 \cdot \exp(-(0.5218)^2 \cdot Fo)$$

$$Fo = -\frac{1}{(0.5218)^2} \ln 0.4525 = 2.91 \; ; \; t = \frac{L^2 \cdot Fo}{\alpha} = \frac{(0.03)^2 * 2.91}{30.85} 10^6$$

$$\boxed{t = 84.89 s}$$

$$Q_0'' = \rho C_p L (T_i - T_\infty)$$
$$= 8530 \frac{kg}{m^3} \cdot 380 \frac{J}{kgK} \cdot 0.03 m (25-700) = -65638350 J/m^2$$

$$\theta^*(0) = 1.045 \cdot \exp[-(0.5218)^2 \cdot 2.91] = 0.473$$

$$Q'' = Q_0'' \left[1 - \frac{\sin(0.5218)}{0.5218} * 0.473\right] = 0.548 \cdot Q_0''$$

$$\boxed{Q'' = -35981239.6 J/m^2}$$

Example problem:

A steel sphere ($\rho = 8000 kg/m^3$, $Cp = 480 J/kgK$, $k = 20 W/mK$, $\alpha = 3{,}9x10^{-6} m^2/s$) of diameter 3 cm is initially at **450 °C**. It is subjected to a two step cooling process. At step 1, steel sphere is cooled in air at **15 °C** with a convection heat transfer coefficient of **25 W/m^2K** until the center temperature reaches **350 °C**. Second step is initiated immediately and steel sphere is cooled in a well-stirred water bath at **20 °C** with a convection heat transfer coefficient of **1200 W/m^2K** until the center temperature reaches **40 °C**. Calculate; **(a)** the surface temperature of steel sphere at the end of the first step and the time required for the cooling process in air and **(b)** the surface temperature of the steel sphere at the end of the second step and the time required for the cooling process in water bath.

$r_0 = 1.5 cm = 0.015 m$ I) $Bi = \frac{h r_0}{3k} = \frac{25 * 0.015}{3 * 20} = 0.00625 < 0.1$

a) $\frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo)$, Since $Bi < 0.1$ $\boxed{T(r_0) = T(0) = 350°C}$

the time required: $\ln \frac{T - T_\infty}{T_i - T_\infty} = -Bi \cdot Fo$

$$Fo = -\frac{1}{Bi} \ln \frac{T - T_\infty}{T_i - T_\infty} = -\frac{1}{0.00625} \ln \frac{350-15}{450-15} = 41.79$$

$$Fo = \frac{\alpha t}{\left(\frac{r_0}{3}\right)^2} = \frac{9 \alpha t}{r_0^2} \; ; \; t = \frac{r_0^2 Fo}{9 \alpha} = \frac{(0.015)^2 \cdot 41.79}{9 * 3.9} * 10^6$$

$$\boxed{t = 267.88 s}$$

II) b) $T_i = 350°C$ $\quad Bi = \frac{1200 * 0.015}{3 * 20} = 0.3 > 0.1$ One-term solution

For sphere $Bi^* = \frac{h r_0}{k} = \frac{1200 * 0.015}{20} = 0.9$; $\varphi_1 = 1.5044$, $C_1 = 1.2488$

Center temperature is: $\theta^*(0) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = C_1 \exp(-\varphi_1^2 Fo)$

$$\frac{1}{C_1} \frac{T(0,t) - T_\infty}{T_i - T_\infty} = \exp(-\varphi_1^2 \cdot Fo)$$

$$-\frac{1}{\varphi_1^2} \ln\left[\frac{1}{C_1} \frac{T(0,t) - T_\infty}{T_i - T_\infty}\right] = Fo \text{ ; } Fo = -\frac{1}{(1.5044)^2} \cdot \ln\left[\frac{1}{1.2488} \frac{40-20}{350-20}\right]$$

$$Fo = 1.336 = \frac{\alpha t}{r_0^2}$$

$$t = \frac{r_0^2 Fo}{\alpha} = \frac{(0.015)^2 * 1.336}{3.9} 10^6$$

$$\boxed{t = 77 s}$$

The surface temperature $T(r_0, t) = ?$ $\quad r^* = \frac{r_0}{r_0} = 1$

$$\theta^*(1) = C_1 \exp(-\varphi_1^2 Fo) \cdot \frac{1}{\varphi_1 r^*} \sin(\varphi_1 r^*)$$

$$= 1.2488 \exp\left[-(1.5044)^2 \cdot 1.336\right] \frac{1}{1.5044 * 1} \sin(1.5044 * 1)$$

$$\theta^*(1) = 0.04027 = \frac{T(r_0) - T_\infty}{T_i - T_\infty}$$

$$T(r_0) = T_\infty + (T_i - T_\infty)\, \theta^*(1)$$

$$= 20 + (350 - 20) \cdot 0.04027$$

$$\boxed{T(r_0) = 33.3°C}$$