

CHAPTER 2. VECTOR FORCES

2.1. Vector

Many physical quantities in engineering mechanics are measured using either scalars or vectors.

Scalar. A scalar is any positive or negative physical quantity that can be completely specified by its magnitude. Examples of scalar quantities include length, mass, and time. Only one number is required to represent a scalar in any dimensional space.

Vector. A vector is any physical quantity that requires both a magnitude and a direction for its complete description. Examples of vectors encountered in statics are force, position, and moment. A vector requires n number of coordinates to be characterized in a n dimensional space. For example, 3 coordinates are required to define a vector in a 3D.

Vector quantities are represented by simply drawing an arrow above it, \vec{A} , or a small line under it, \underline{A} . In this course, the vectors will be presented with an arrow above them, such as \vec{A} . Magnitude of a vector can be expressed simply by vector itself without an arrow, A or the vector representation can be written in an absolute value sign, $|\vec{A}|$

Vectors can be divided into three categories corresponding to their physical representations.

- i. **Floating Vector (TR: Kayan Vektör)** If a vector can move on the line of action from one point to another provided that it maintains its magnitude and direction, it is called floating vector. Force is an example of floating vector in Statics.

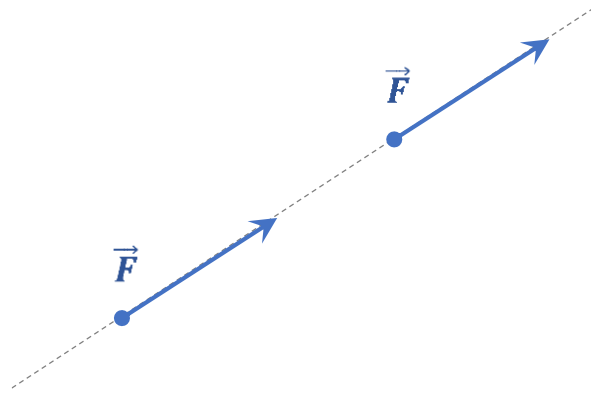


Fig. 2.1. Floating vector

Two floating vectors on the same line of action are equal to each other provided that both have the same magnitude and direction even if their point of actions are not the same.

ii. **Free Vector (TR: Serbest Vektör)** If the line of action does not change the effect of a vector provided that the vector has same magnitude and direction, it can move freely in the space and is called free vector. Moment is an example of free vector in Statics.

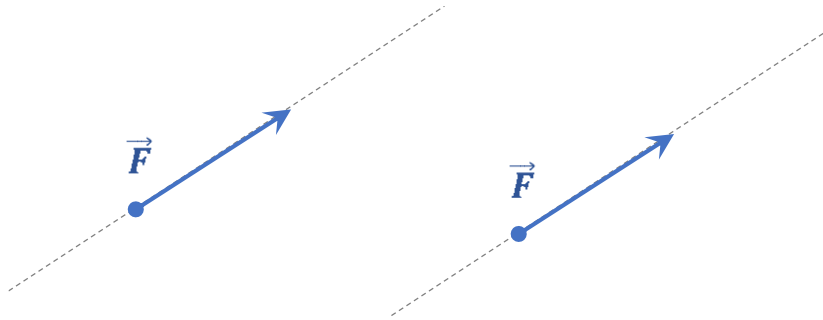


Fig. 2.2. Free vector

Two free vectors are equal to each other provided that both has same magnitude and direction.

iii. **Fixed Vector (TR: Bağlı Vektör)** If not only magnitude and direction but also point of application change the effect of a vector, it is called fixed vector and its position cannot be changed. Force is an example of fixed vector in deformable-body mechanics. Two fixed vectors are equal to each other provided that both have same magnitude, direction and same point of application.

2.2. Vector Operations

2.2.1. Multiplication and Division of a Vector by a Scalar

If a vector is multiplied by a positive scalar, its magnitude is increased by that amount. Multiplying by a negative scalar will also change the directional sense of the vector.

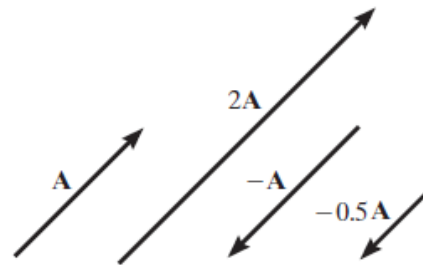


Fig. 2.3. Scalar multiplication and division

2.2.2. Vector Addition

Two vectors can be added using the parallelogram law of addition and obtained **resultant vector**. Addition procedure:

- First join the tails of the vectors (let's name them as \vec{A} and \vec{B}) at a point O to make them concurrent.
- From the head of first vector, \vec{A} , draw a line parallel to the second vector, \vec{B} .
- Draw another line from the head of \vec{B} that is parallel to \vec{A} .
- These two lines intersect at point P to form the adjacent sides of a parallelogram.
- The diagonal of this parallelogram from O to P forms R, which then represents the resultant vector $\vec{R} = \vec{A} + \vec{B}$.

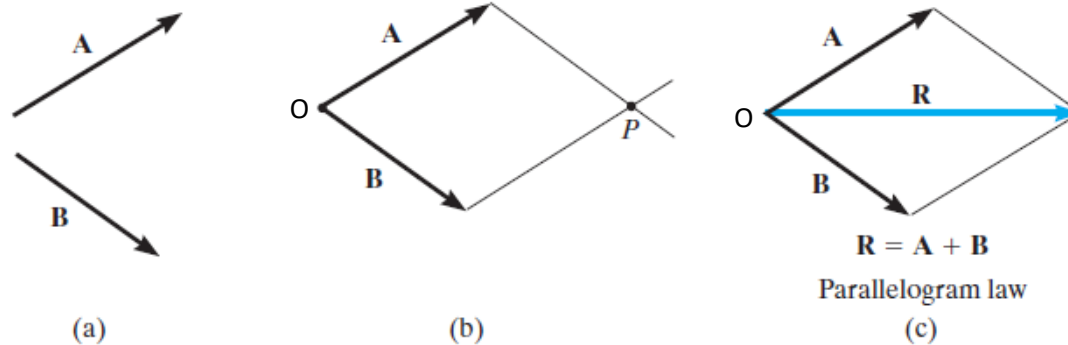
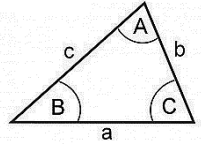


Fig. 2.4. Addition of two vectors

The order of \vec{A} and \vec{B} does not change the resultant vector. Thus, vector addition is commutative, i.e. $\vec{R} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$.

The magnitude and direction of the resultant can be obtained using sine and cosine theorem.



For the given triangle, cosine theorem can be written as $a^2 = b^2 + c^2 - 2bc \cos A$ whereas sine theorem can be expressed as

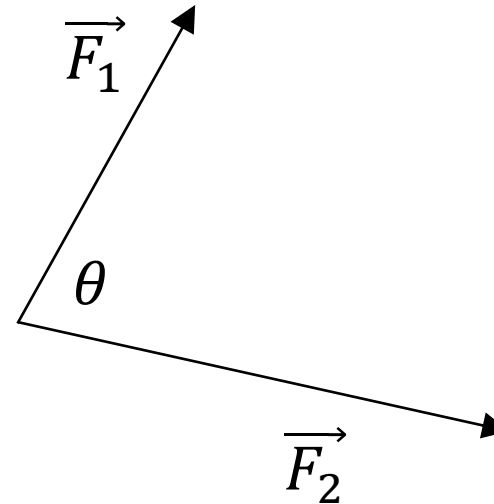
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. Applying these theorems to *OPL* triangle, following expressions can be obtained.

$$R = \sqrt{F_1^2 + F_2^2 - 2F_1F_2 \cos(180 - \theta)}, \quad \cos(180 - \theta) = -\cos \theta$$

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos(\theta)}$$

$$\frac{R}{\sin(180 - \theta)} = \frac{F_1}{\sin \alpha} = \frac{F_2}{\sin(\theta - \alpha)}, \quad \sin(180 - \theta) = \sin \theta$$

$$\frac{R}{\sin \theta} = \frac{F_1}{\sin \alpha} = \frac{F_2}{\sin(\theta - \alpha)}$$



Equations have specific expressions for $\theta = 0$, $\theta = 90$ and $\theta = 180$.

i. $\theta = 0$ (Both vectors have same directions)

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2(1)} = F_1 + F_2$$

$$\alpha = \tan^{-1} \left(\frac{F_1(0)}{F_1(1) + F_2} \right) = 0$$

If the two vectors have same direction, *i.e.* $\theta = 0$, the resultant force can be algebraic or scalar addition of two vectors and the resultant force is on the same direction with vectors added, *i.e.* $\alpha = 0$.

ii. $\theta = 90$ (vectors are perpendicular to each other)

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2(0)} = \sqrt{F_1^2 + F_2^2}$$

$$\alpha = \tan^{-1} \left(\frac{F_1(1)}{F_1(0) + F_2} \right) = \tan^{-1} \left(\frac{F_1}{F_2} \right)$$

iii. $\theta = 180$ (Vectors have opposite directions)

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2(-1)} = F_1 - F_2$$

$$\alpha = \tan^{-1} \left(\frac{F_1(0)}{F_1(-1) + F_2} \right) = 0$$

If the two vectors have opposite direction, i.e. $\theta = 180$, the resultant force can be algebraic or scalar subtraction of two vectors and the resultant force is on the same direction with the vector which is the one from the added vectors that has the bigger magnitude, i.e. $\alpha = 0$ or $\alpha = 180$.

Addition of Several Vectors

If more than two vectors are to be added, successive applications of the parallelogram law can be carried out in order to obtain the resultant force. For example, if three forces \vec{F}_1 , \vec{F}_2 and \vec{F}_3 act at a point O, the resultant of any two of the forces is found, say $\vec{F}_1 + \vec{F}_2$, and then this resultant is added to the third force, yielding the resultant of all three forces; i.e., $\vec{R} = (\vec{F}_1 + \vec{F}_2) + \vec{F}_3$. Using the parallelogram law to add more than two forces, as in the example, often requires extensive geometric and trigonometric calculation to determine the numerical values for the magnitude and direction of the resultant. Instead, problems of this type are easily solved by using the **rectangular component method** which will be explained in this chapter.

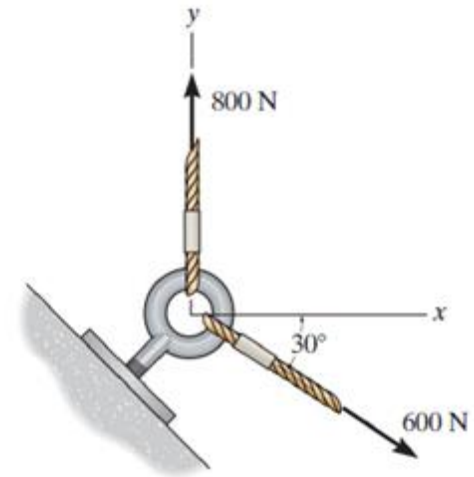
2.2.3. Vector Subtraction

The subtraction of two forces (let's name them as \vec{A} and \vec{B}) can be expressed as $\vec{R} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$. Therefore, rules of vector addition can be applied to vector subtraction.

NOTE: Since force is a vector, all the operations can be applied to the forces.

Example 2.1. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.

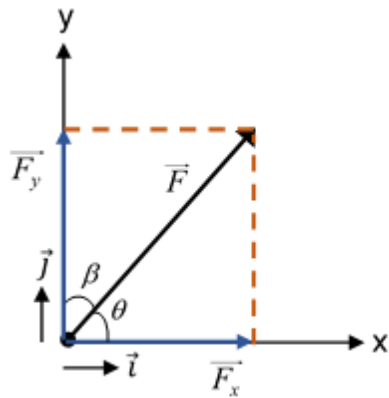
Solution:



2.3. Addition of a System of Coplanar Forces

2.3.1. Rectangular Components of a Vector

When a force is resolved into two components perpendicular to each other, e.g. as in a Cartesian coordinate system, the components are then called **rectangular components**.



$$F_x = F \cos \theta \text{ or } F_x = F \sin \beta$$

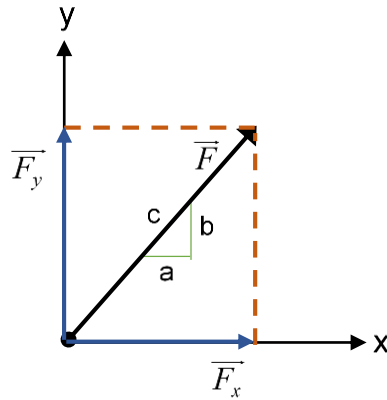
$$F_y = F \sin \theta \text{ or } F_y = F \cos \beta$$

It is also possible to represent the x and y components of a force in terms of Cartesian unit vectors \vec{i} and \vec{j} . They are called **unit vectors** because they have a dimensionless magnitude of 1, and so they can be used to designate the directions of the x and y axes, respectively. Thus, **Cartesian vector notation** of a vector is given below.

$$\vec{F}_x = F_x \vec{i}, \quad \vec{F}_y = F_y \vec{j}$$

$$\vec{F} = \vec{F}_x + \vec{F}_y = F_x \vec{i} + F_y \vec{j}$$

If a vector is given by a slope representation as in the below figure, the rectangular components can be found as follows.

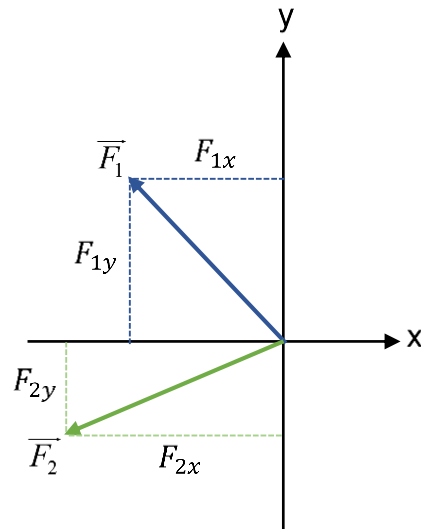


$$a \perp b \rightarrow c^2 = a^2 + b^2$$

$$\frac{b}{c} = \frac{F_y}{F} \rightarrow F_y = F \left(\frac{b}{c} \right)$$

$$\frac{a}{c} = \frac{F_x}{F} \rightarrow F_x = F \left(\frac{a}{c} \right)$$

It must be noted that, the signs of the components should be determined by the direction of the force.



$$\vec{F}_1 = -F_{1x}\vec{i} + F_{1y}\vec{j}$$

$$\vec{F}_2 = -F_{2x}\vec{i} - F_{2y}\vec{j}$$

2.3.2. Resultant of Coplanar Forces

We can use rectangular components to determine the resultant of several coplanar forces, *i.e.*, forces that all lie in the same plane, *e.g.* *xy* plane. To do this, each force is first resolved into its *x* and *y* components, and then the respective components are added using scalar algebra since they are collinear. The resultant force is then formed by adding the resultant components using the parallelogram law.

$$\vec{F}_1 = F_{1x}\vec{i} + F_{1y}\vec{j}$$

$$\vec{F}_2 = F_{2x}\vec{i} + F_{2y}\vec{j}$$

$$\vdots$$

$$\vec{F}_n = F_{nx}\vec{i} + F_{ny}\vec{j}$$

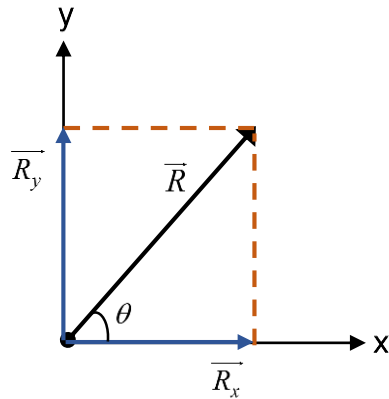
$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n$$

$$\vec{R} = (F_{1x}\vec{i} + F_{1y}\vec{j}) + (F_{2x}\vec{i} + F_{2y}\vec{j}) + \cdots + (F_{nx}\vec{i} + F_{ny}\vec{j})$$

$$\vec{R} = (F_{1x} + F_{2x} + \cdots + F_{nx})\vec{i} + (F_{1y} + F_{2y} + \cdots + F_{ny})\vec{j}$$

$$\vec{R} = R_x\vec{i} + R_y\vec{j}$$

$$R_x = \sum_{i=1}^n F_{ix}, \quad R_y = \sum_{i=1}^n F_{iy}$$



Once the components of resultant force are determined, they may be sketched along the x and y axes with their proper sense of direction, and the resultant force can be determined from vector addition.

$$R = \sqrt{R_x^2 + R_y^2}$$

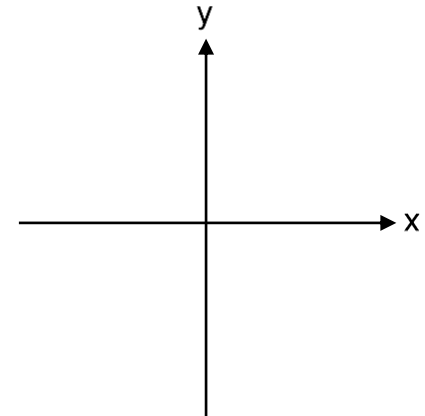
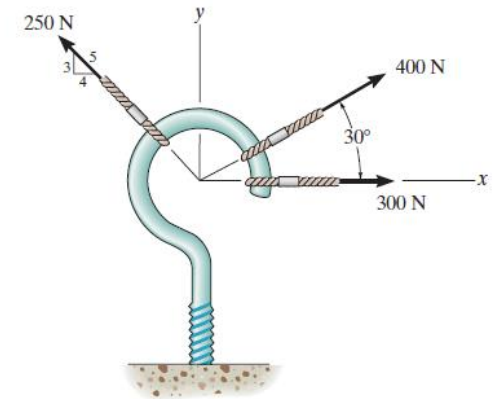
$$\tan \theta = \frac{R_y}{R_x} \rightarrow \theta = \tan^{-1} \left(\frac{R_y}{R_x} \right)$$

Resultant Force Problems

- i. Resolve all the given forces into their components
- ii. Find components of resultant simply using equations, $R_x = \sum_{i=1}^n F_{ix}$, $R_y = \sum_{i=1}^n F_{iy}$
- iii. Calculate magnitude and angle of the resultant

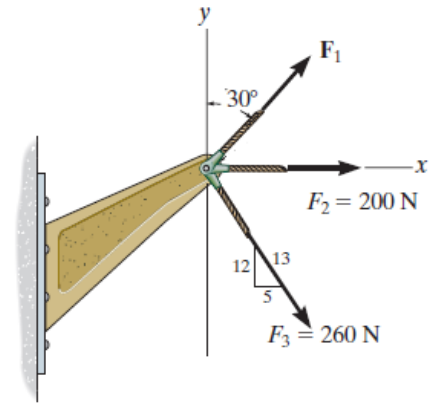
Example 2.2

Determine the magnitude and direction of the resultant force.



Example 2.3.

If the resultant force acting on the bracket is in the $+x$ direction, determine the magnitudes of \vec{F}_1 and the resultant force.

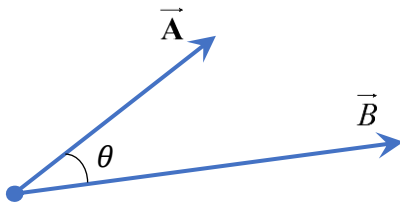


2.4.5. Dot (Scalar) Product

The **dot product** of two vectors \vec{A} and \vec{B} is defined as the product of the magnitudes of A and B and of the cosine of the angle θ formed by \vec{A} and \vec{B} . The dot product of \vec{A} and \vec{B} is denoted by $\vec{A} \cdot \vec{B}$.

$$\vec{A} \cdot \vec{B} = AB \cos \theta$$

The result of dot product of two vectors is scalar, therefore dot product is also called **scalar product**.



Laws of operation

- i. Commutative law: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- ii. Multiplication by a scalar: $m(\vec{A} \cdot \vec{B}) = (m\vec{A}) \cdot \vec{B} = \vec{A} \cdot (m\vec{B})$
- iii. Distributive law: $\vec{C} \cdot (\vec{A} + \vec{B}) = \vec{C} \cdot \vec{A} + \vec{C} \cdot \vec{B}$

The dot products of vectors can be expressed in terms of rectangular components.

$$\vec{A} \cdot \vec{B} = (A_x \vec{i} + A_y \vec{j} + A_z \vec{k}) \cdot (B_x \vec{i} + B_y \vec{j} + B_z \vec{k})$$

In dot product of unit vectors is equal to 1 if unit vectors are same and is equal to 0 if unit vectors are different from the definition of dot product. Some of these dot products are given as an example.

$$\vec{i} // \vec{i} \rightarrow \vec{i} \cdot \vec{i} = ii \cos 0 = 1$$

$$\vec{i} \perp \vec{j} \rightarrow \vec{i} \cdot \vec{j} = ij \cos 90 = 0$$

$$\vec{i} \perp \vec{k} \rightarrow \vec{i} \cdot \vec{k} = ik \cos 90 = 0$$

Putting dot products of unit vectors into dot product of $\vec{A} \cdot \vec{B}$, the product turns into a very simple expression.

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

In the particular case when A and B are equal, we note that

$$\vec{A} \cdot \vec{A} = A_x A_x + A_y A_y + A_z A_z = A_x^2 + A_y^2 + A_z^2 = A^2$$

Application

The angle formed between two vectors or intersecting lines

The dot product of \vec{A} and \vec{B} which have formed by an angle θ is given below.

$$\vec{A} \cdot \vec{B} = AB \cos \theta = A_x B_x + A_y B_y + A_z B_z$$

If the angle is drawn from this equations, θ becomes,

$$\cos \theta = \frac{A_x B_x + A_y B_y + A_z B_z}{AB}$$

and, θ can be calculated from this equality.