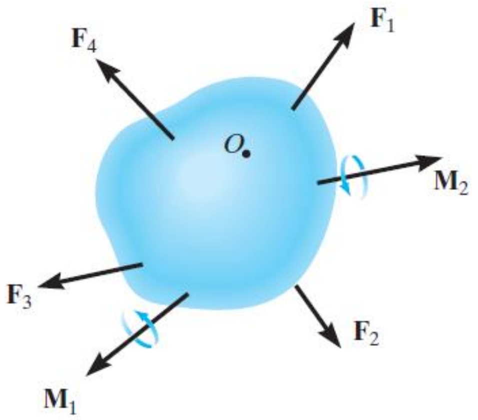
CHAPTER 5. EQUILIBRIUM OF A RIGID BODY

5.1. Conditions for Rigid-Body Equilibrium



This body is subjected to an external force and couple moment system that is the result of the effects of gravitational, electrical, magnetic, or contact forces caused by adjacent bodies. The internal forces caused by interactions between particles within the body are not shown in this figure because these forces occur in equal but opposite collinear pairs and hence will cancel out. Using the methods of the previous chapter, the force and couple moment system acting on a body can be reduced to an equivalent resultant force and resultant couple moment at any arbitrary point O on or off the body. If this resultant force and couple moment are both equal to zero, then the body is said to be in **equilibrium**. Mathematically, the equilibrium of a body is expressed as,

$$\vec{\boldsymbol{F}}_R = \sum \vec{\boldsymbol{F}} = \mathbf{0}$$

$$(\boldsymbol{M}_R)_O = \sum \boldsymbol{M}_O = \mathbf{0}$$

These two equations are not only necessary for equilibrium, they are also sufficient.

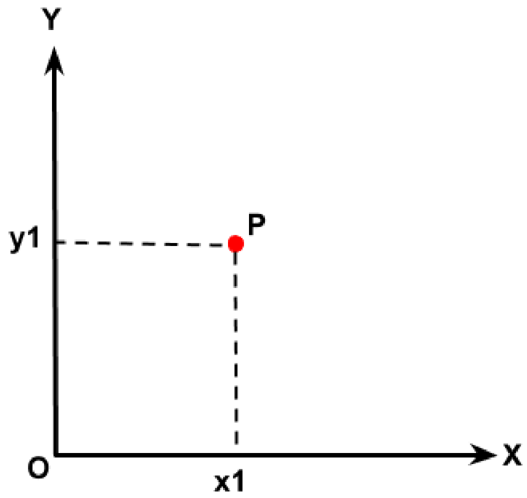
When applying the equations of equilibrium, we will assume that the body remains rigid. In reality, however, all bodies deform when subjected to loads. Although this is the case, most engineering materials such as steel and concrete are very rigid and so their deformation is usually very small. Therefore, when applying the equations of equilibrium, we can generally assume that the body will remain rigid and not deform under the applied load without introducing any significant error. This way the direction of the applied forces and their moment arms with respect to a fixed reference remain the same both before and after the body is loaded.

EQUILIBRIUM IN TWO DIMENSIONS

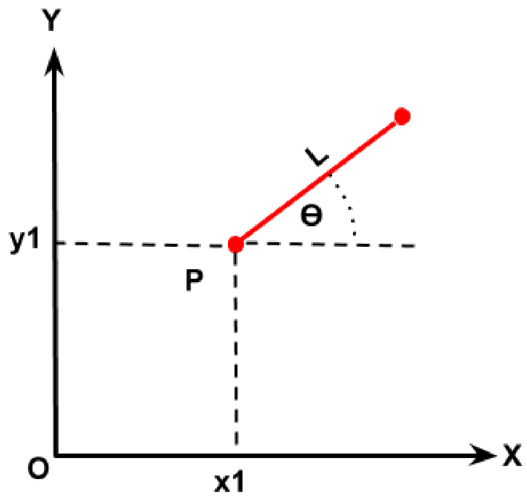
We will first consider the case where the force system acting on a rigid body lies in or may be projected onto a single plane and, furthermore, any couple moments acting on the body are directed perpendicular to this plane. This type of force and couple moment system is often referred to as a two-dimensional or coplanar force system.

5.2. Degree of Freedom

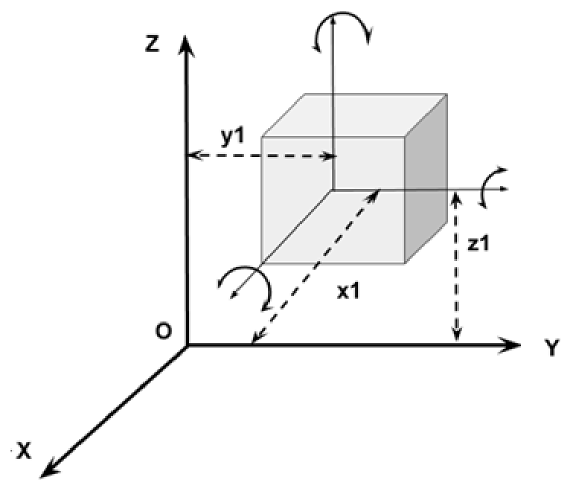
Degree-of-freedom (DOF) is defined as the minimum number of independent variables required to define the position of a rigid body in space. In other words, DOF defines the number of directions that a body can move.



To define/constraint the position of a point P in a plane, only its distance from the origin in x and y-axis is required. Therefore, the point P has two DOFs in a plane.



To define/constraint the position of a line L or a planar rigid body, its distance from the origin in x and y-axis and the angle from x-axis is required. Therefore, the line L or a rigid body has three DOFs in 2D space. The position and orientation of a rigid-body in a plane is defined by two components of translation and one component of rotation, which means that it has three DOFs.



To define/constraint the position of a rigid body in 3D space, its distance from the origin in x, y and z-axis and the angle from xy, xz and yz plane are required. Therefore, a rigid body has six DOFs in 3D space.

The position and orientation of a rigid body in space is defined by three components of translation and three components of rotation, which means that it has six DOFs.

5.3. Free-Body Diagrams

Successful application of the equations of equilibrium requires a complete specification of all the known and unknown external forces that act on the body. The best way to account for these forces is to draw a free-body diagram. This diagram is a sketch of the outlined shape of the body, which represents it as being isolated or "free" from its surroundings, i.e., a "free body." On this sketch it is necessary to show all the forces and couple moments that the surroundings exert on the body so that these effects can be accounted for when the equations of equilibrium are applied. A thorough understanding of how to draw a free-body diagram is of primary importance for solving problems in mechanics.

Before presenting a procedure how to draw a free-body diagram, we will first consider support types encountered in rigid-body equilibrium problems.

- A support prevents the translation of a body in a given direction by exerting a force on the body in the opposite direction.
- A support prevents the rotation of a body in a given direction by exerting a couple moment on the body in the opposite direction.

Since supports/connections prevent the motion of a body or structural member in any given direction, a reaction will occur at the points at which they are applied.

There are three common types of supports or connections: **roller**, **pinned** and **fixed**. All of these supports can be located anywhere along a structural element, i.e., at the ends, at midpoints, or at any other intermediate points.

i. **Roller**

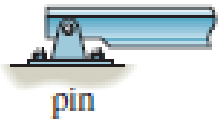
Roller supports are free to rotate and translate along the surface on which the roller rests. The surface can be horizontal, vertical or sloped at any angle.

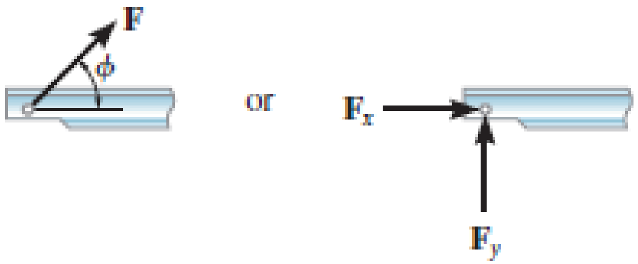
The resulting reaction force is always a single force that is perpendicular to and away from the surface. As a result, only one unknown force is in roller support.

Roller supports are commonly located at one end of long bridges. This allows the bridge structure to expand and contract with temperature changes. The expansion forces could fracture the supports at the banks if the bridge structure was “locked” in place.

Roller supports can also take the form of rubber bearings, rockers, or a set of gears which are designed to allow a limited amount of lateral movement.

ii. Pinned





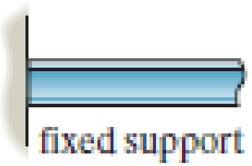
A pinned support can resist both vertical and horizontal forces but not a moment. They will allow the structural member to rotate, but not to translate in any direction.

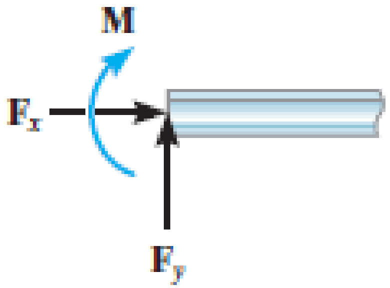
The representation of a pinned support includes both horizontal and vertical forces. As a result, two unknown forces in a pin support.

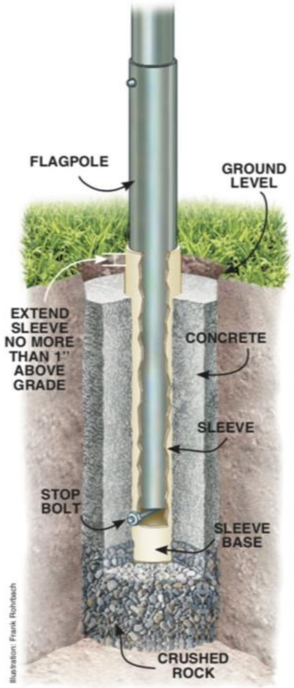
Many connections are assumed to be pinned even though they might resist a small amount of moment in reality. It is also true that a pinned connection could allow rotation in only one direction; providing resistance to rotation in any other direction. The knee can be idealized as a connection which allows rotation in only one direction and provides resistance to lateral movement.

A single pinned connection is usually not sufficient to make a structure stable. Another support must be provided at some point to prevent rotation of the structure.

iii. Fixed



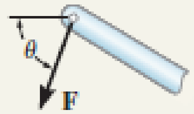
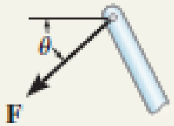
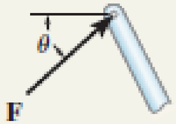
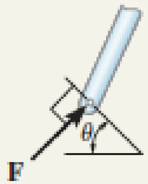
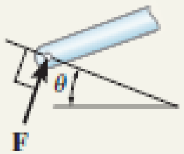


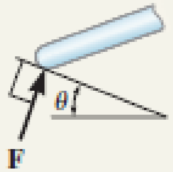
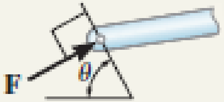
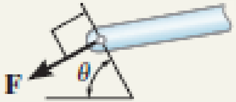
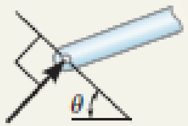
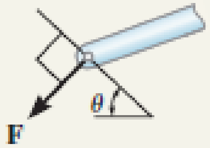


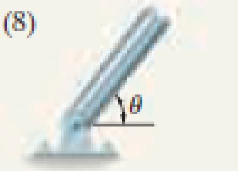
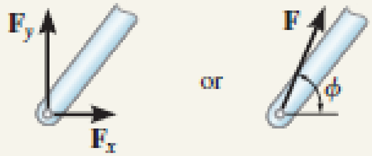
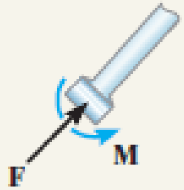
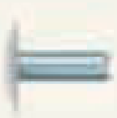
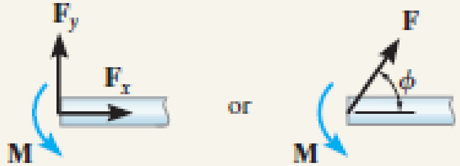
Fixed supports can resist vertical and horizontal forces as well as a moment. Since they restrain both rotation and translation, they are also known as rigid supports. This means that a structure only needs one fixed support in order to be stable. All three equations of equilibrium can be satisfied.

A flagpole set into a concrete base is a good example of this kind of support.

The representation of fixed supports always includes two forces (horizontal and vertical) and a moment (three unknown forces).

Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.
(2)  weightless link	 or 	One unknown. The reaction is a force which acts along the axis of the link.
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4)  rocker		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.

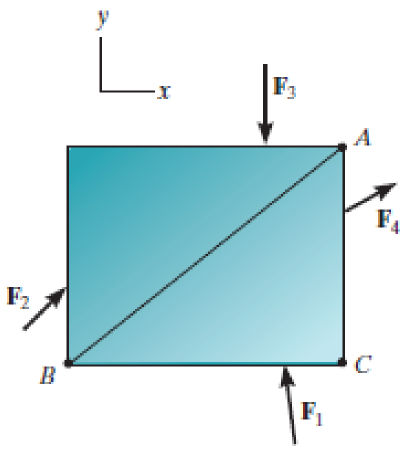
(5)  smooth contacting surface		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(6)  roller or pin in confined smooth slot	 or 	One unknown. The reaction is a force which acts perpendicular to the slot.
(7)  member pin connected to collar on smooth rod	 or 	One unknown. The reaction is a force which acts perpendicular to the rod.

(8)  smooth pin or hinge		Two unknowns. The reactions are two components of force, or the magnitude and direction ϕ of the resultant force. Note that ϕ and θ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].
(9)  member fixed connected to collar on smooth rod		Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod.
(10)  fixed support		Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction ϕ of the resultant force.

NOTES:

- No equilibrium problem should be solved without first drawing the free-body diagram, so as to account for all the forces and couple moments that act on the body.
- If a support prevents translation of a body in a particular direction, then the support, when it is removed, exerts a force on the body in that direction.
- If rotation is prevented, then the support, when it is removed, exerts a couple moment on the body.
- Internal forces are never shown on the free-body diagram since they occur in equal but opposite collinear pairs and therefore cancel out.
- The weight of a body is an external force, and its effect is represented by a single resultant force acting through the body's center of gravity ***G***.
- Couple moments can be placed anywhere on the free-body diagram since they are free vectors. Forces can act at any point along their lines of action since they are sliding vectors.

5.4. Equations of equilibrium



When a body is subjected to a system of coplanar forces, which all lie in the xy plane, the forces can be resolved into their x and y components. Consequently, the conditions for equilibrium in two dimensions are

$$\sum \boldsymbol{F_x} = \mathbf{0}, \quad \sum \boldsymbol{F_y} = \mathbf{0}, \quad \sum \boldsymbol{M_O} = \mathbf{0}$$

Here $\sum \boldsymbol{F_x}$ and $\sum \boldsymbol{F_y}$ represent, respectively, the algebraic sums of the x and y components of all the forces acting on the body, and $\sum \boldsymbol{M_O}$ represents the algebraic sum of the couple moments and the moments of all the force components about the z axis, which is perpendicular to the x–y plane and passes through the arbitrary point O.

Although above equations are most often used for solving coplanar equilibrium problems, two alternative sets of three independent equilibrium equations may also be used.

$$\sum \boldsymbol{F_x} = \mathbf{0}, \quad \sum \boldsymbol{M_A} = \mathbf{0}, \quad \sum \boldsymbol{M_B} = \mathbf{0}$$

When using these equations it is required that a line passing through points A and B is not parallel to the y axis

$$\sum \boldsymbol{M_A} = \mathbf{0}, \quad \sum \boldsymbol{M_B} = \mathbf{0}, \quad \sum \boldsymbol{M_C} = \mathbf{0}$$

Here it is necessary that points A, B, and C do not lie on the same line.

Procedure of Analysis

i. Establish the x and y axis in any suitable orientation. Choosing a good orientation makes the solution simpler.

ii. Remove all supports and draw FBD of the body. Show all the forces and couple moments acting on the body. Label all the loadings and specify their directions relative to the x or y axis. The sense of a force or couple moment having an unknown magnitude but known line of action can be assumed.

iii. Indicate the dimensions of the body necessary for computing the moments of forces.

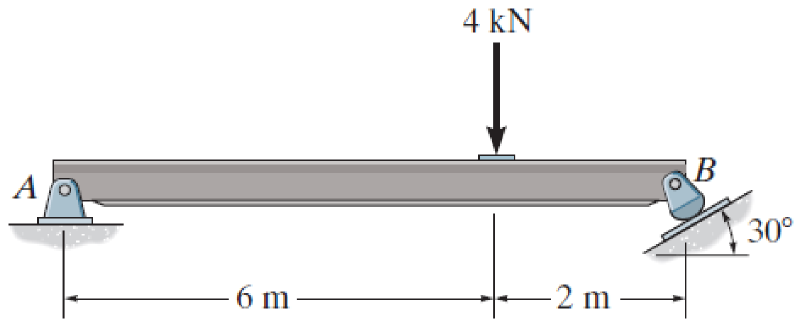
iv. Apply the moment equation of equilibrium, $\sum \boldsymbol{M_O} = \mathbf{0}$, about a point (*O*) that lies at the intersection of the lines of action of two unknown forces. In this way, the moments of these unknowns are zero about *O*, and a direct solution for the third unknown can be determined.

v. When applying the force equilibrium equations, $\sum \boldsymbol{F_x} = \mathbf{0}$ and $\sum \boldsymbol{F_y} = \mathbf{0}$, orient the *x* and *y* axes along lines that will provide the simplest resolution of the forces into their *x* and *y* components.

vi. If the solution of the equilibrium equations yields a negative scalar for a force or couple moment magnitude, this indicates that the sense is opposite to that which was assumed on the free-body diagram.

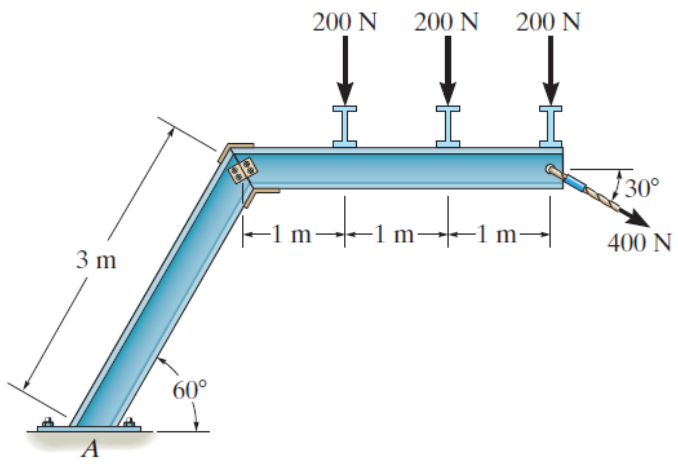
Example 1.

Determine the horizontal and vertical components of reaction at the pin A and the reaction of the rocker B on the beam.



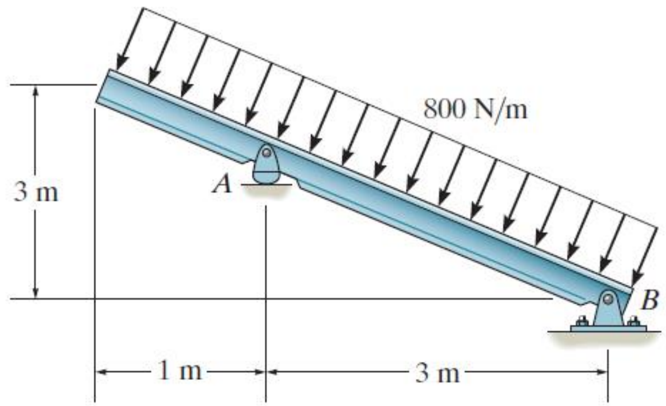
Example 2.

Determine the components of reaction at the fixed support A. Neglect the thickness of the beam.



Example 3.

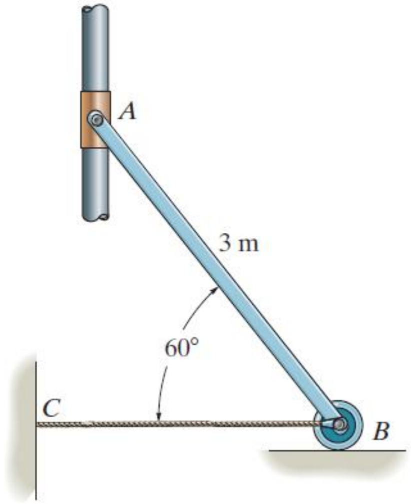
Determine the reactions at the supports.



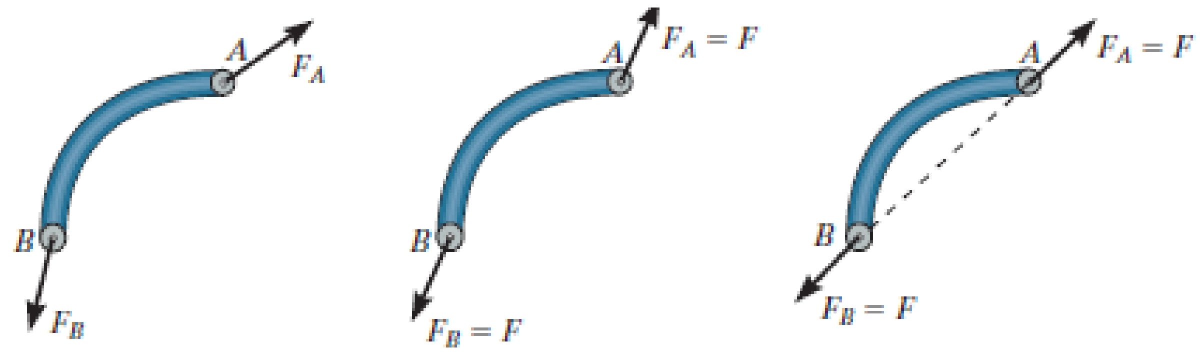
Example 4.

a) The uniform rod AB has a mass of 40 kg. Determine the forces in the cable and supports when the rod is in the position shown. There is a smooth collar at A.

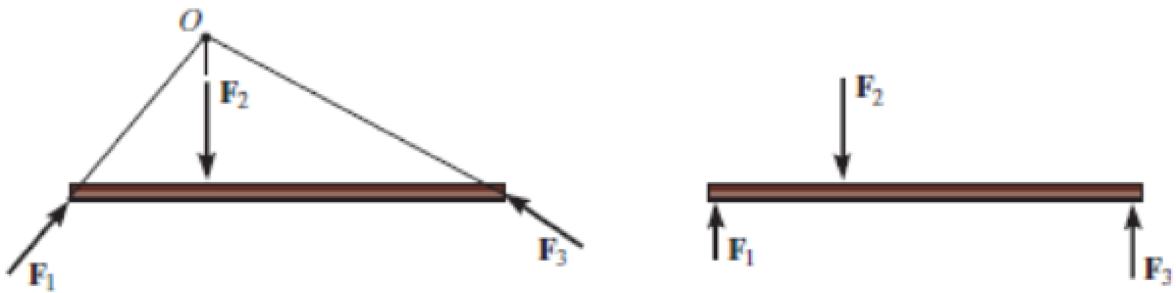
b) If the cable CB can sustain a maximum load of 1500 N before it fails, determine the maximum weight of the rod.



5.5. Two and Three Force Members



For any **two-force member** to be in equilibrium, the two forces acting on the member must have the same magnitude, act in opposite directions, and have the same line of action, directed along the line joining the two points where these forces act.



If a member is subjected to only three forces, it is called a **three-force member**. Moment equilibrium can be satisfied only if the three forces form a concurrent or parallel force system. To illustrate, consider the member subjected to the three forces $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_2$, shown in the figure. If the lines of action of $\vec{F}_1$ and $\vec{F}_2$ intersect at point O, then the line of action of $\vec{F}_3$ must also pass through point O so that the forces satisfy $\sum \vec{M}_O$ = 0.

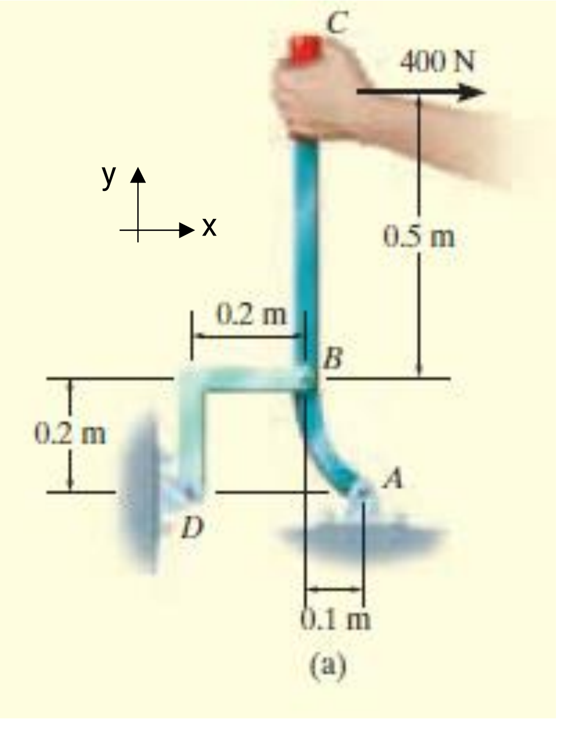
As a special case, if the three forces are all parallel, the location of the point of intersection, O, will approach infinity.

Example 5.

The lever ABC is pin supported at A and connected to a short link BD as shown in the figure. If the weight of the members is negligible, determine the force of the pin on the lever at A.

$$\sin\theta = 0.496$$

$$\cos\theta = 0.868$$



Example 6.

The beam of negligible weight is supported horizontally by two springs. If the beam is horizontal and the springs are unstretched when the load is removed, determine the angle of tilt of the beam when the load is applied.