ON FRACTION MATRICES $\left(\frac{A}{B}\right)$ AND EIGENVALUES-EIGENVECTORS

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ABSTRACT

This study is about division in matrices and eigenvalues-eigenvectors. The concept of eigenvalues-eigenvectors in the literature is discussed. The status of division operation on these concepts is analyzed. The eigenvalues and eigenvectors of the elements forming the division are compared with the result. The matrix resulting and the resulting difference are investigated. The eigenvalues-eigenvectors of the constituent matrices the division and the eigenvalues-eigenvectors obtained from the result matrix are examined. The brief literature review of the study is written and the summary history of this study is added in the first section. The theorems related to the subject are listed. Some applications related to this topic are given. The preliminary information that will form the second part is given. The new developments and findings are investigated in the next section. The changes of known definitions, theorems and lemmas are observed. If there are unprotected cases, examples are given for these situations. The new contributions on the parallel cases of matrix product and scalar product in the eigenvalue definition are investigated. The important hints that will contribute to transformations are obtained. The concept of rotation in the planes in the studies is concluded carried to higher dimensions with this contribution. The rotation in the plane is realized only in two directions. There is no direction limit in dimension 3. This situation covers matrices larger than order 3rd. The computation of parallel states corresponding to the same eigenvalue is expected to be of new interest. In short, the study marks the beginning of innovations between multiplication-division and eigenvalue-eigenvector.

Keywords : division, eigenvalues, eigenvectors, multiplication.

1. INTRODUCTION

Eigenvalues and eigenvectors are the foundation of matrix algebra, systematic structures and geometry. In the 18th century, Leonhard Euler realized that rotation and Joseph-Louis Lagrange realized that the principal axes of momentum and inertia are eigenvectors. Augustin-Louis

Cauchy established and generalized the relation between quadratic equations and surfaces in the year 1800. This subject is the subject of study of many famous mathematicians and practitioners until today. They are Charles Hermite, Francesco Brioschi, Joseph Liouville and David Hilbert.

Eigenvalues and eigenvectors are very useful in the motion of solids, sound, image and structure of atomic. They are also used in navigation systems today.
Eigenvectors are the structure vector of parallel. There is the data matrix and vectors on one side of the equation. There are parallel vectors on the other side of the equation.

Let F is a field. The set of matrices over F is denoted by $M_n(F)=\left\{\left[a_{ij}\right]_n \middle| a_{ij}\in F, n\in\mathbb{Z}^+\right\}$ and the set of regular matrices over F is denoted by

$$R_n(F)=\left\{\left[a_{ij}\right]_n \middle| a_{ij}\in F, n\in\mathbb{Z}^+\right\}.$$

Let $A\in M_n(F)$. The trace of a square matrix A is the number obtained by summing the diagonal entries of A. It is denoted by $Tr(A)$. That is,

$$Tr(A)=\sum_{i=1}^{n} a_{ii}.$$

The transpose of $A\in M_n(F)$ is denoted by A^T.

Mathematically, $\vec{u}_\lambda$ is an eigenvector and λ the corresponding eigenvalue of a transformation T if the equation $T(\vec{u}_\lambda)=\lambda\vec{u}_\lambda$, then, T and $\vec{u}_\lambda$ can be represented relative to that basis by a matrix A— a two-dimensional array—and respectively a column vector $\vec{u}_\lambda$—a one-dimensional vertical array.
Computing eigenvalues and eigenvectors of matrices:

$$A\vec{u}_\lambda=\lambda\vec{u}_\lambda \Rightarrow A\vec{u}_\lambda-\lambda\vec{u}_\lambda=0$$
$$(A-\lambda I_n)\vec{u}_\lambda=0$$
$$p_A(\lambda)=\sum_{k=0}^{n}\lambda^{n-k}(-1)^k\chi_A^k, \text{ where } \chi_A^k=\frac{1}{k!}\begin{vmatrix} tr(A) & k-1 & 0 & \cdots & 0 \\ tr(A^2) & tr(A) & k-2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ tr(A^{k-1}) & tr(A^{k-2}) & & \cdots & 1 \\ tr(A^k) & tr(A^{k-1}) & & \cdots & tr(A) \end{vmatrix}.$$

Lemma 1.1. Let $A,B\in M_n(F)$. Then, the solution of the linear matrix equation $XA=B$ is

$$X=\left(\frac{B^T}{A^T}\right)^T.$$

Proposition 1.2. Let $A, B \in M_n(F)$. If the factors of matrix A is BA_1 and the factors of matrix B is AB_1 then

ii. The rational matrix $\frac{A}{B}$ is equal to matrix A_1.

(ii) The rational matrix $\frac{A}{B}$ is equal to matrix $\frac{I_n}{B_1}$.

Theorem 1.3. Let $A, B, X \in M_n(F)$ and X unknowns matrix. Then, in the solution of the equation $AX = B$, there are regular matrices $A = B_2A_3$, $B = B_2B_3$ such as B_2, B_3 and A_3, and the rational matrix $\frac{B_3}{A_3}$ is the solution of the equation $AX = B$. This solution is equal to the rational matrix $\frac{B}{A}$.

The information given so up to this point is the data studied. The subject of embedding another matrix in a regular matrix is interesting.

2. FRACTION MATRICES AND EIGENVALUES-EIGENVECTORS

This section is about the eigenvalues and eigenvectors of matrices A and B and the eigenvalues of the fraction matrix $\frac{A}{B}$. Any regular matrix is written as the division of two regular matrices of the same order. That is,

$$A = \frac{BA}{B} = \frac{C}{B}, \text{ where } A, B, C \in R_n(F).$$

Proposition 2. 1. Let $A, B \in R_n(F)$. The order of the fraction matrices $\frac{A}{B}$ and $\frac{B}{A}$ is n.

Lemma 2.2. Let $A, B \in M_n(F)$. The characteristic polynomials of the fraction matrices $\frac{A}{B}$ and $\frac{B}{A}$ are

i. $p_{\frac{A}{B}}(\lambda) = \sum_{k=0}^{n} \lambda^{n-k} (-1)^k \chi^k_{\frac{A}{B}}$.

ii. $p_{\frac{B}{A}}(\lambda) = \sum_{k=0}^{n} \lambda^{n-k} (-1)^k \chi^k_{\frac{B}{A}}$.

The degrees of two characteristic polynomials $p_{\frac{A}{B}}(\lambda)$ and $p_{\frac{B}{A}}(\lambda)$ are n.

Proposition 2. 3. If the characteristic polynomials of the fraction matrices $\frac{A}{B}$ is $p_{\frac{A}{B}}(\lambda)$, then

$$p_{\frac{A}{B}}(\lambda) = p_{\frac{I_n}{B_1}}(\lambda), \text{ where } B = AB_1.$$

Proof. The proof of the Proposition is clear by the simplification properties.

Theorem 2.4. If $A,B,C \in M_n(F)$, then the following are equal.

i. $p_A(\lambda)=\sum_{k=0}^{n}\lambda^{n-k}(-1)^k \chi_A^k$

ii. $p_{\frac{BA}{B}}(\lambda)=\sum_{k=0}^{n}\lambda^{n-k}(-1)^k \chi_{\frac{BA}{B}}^k$.

iii. $p_{\frac{C}{B}}(\lambda)=\sum_{k=0}^{n}\lambda^{n-k}(-1)^k \chi_{\frac{C}{B}}^k$, where $\frac{C}{B}=A$.

Theorem 2.5. For any $A,B,C \in R_n(F)$ If $BA=C$, then

$$p_B(\lambda)=p_{\left(\frac{C^T}{A^T}\right)^T}(\lambda).$$

Proof. If $BA=C$, then $B=\left(\frac{C^T}{A^T}\right)^T$ by Lemma 1.1. And

$$p_B(\lambda)=p_{\left(\frac{C^T}{A^T}\right)^T}(\lambda).$$

Theorem 2.6. If a matrix $\frac{A}{B}$ has an eigenvalue λ and an eigenvector $\vec{u}_\lambda$. Then there is a matrix B_1 such that eigenvalue λ and eigenvector $\vec{u}_\lambda$ are eigenvalue and eigenvector belong matrix $\frac{I_n}{B_1}$.

Proof. If $\frac{A}{B}\vec{u}_\lambda=\lambda\vec{u}_\lambda$ then

$$\frac{A}{AB_1}\vec{u}_\lambda=\lambda\vec{u}_\lambda \Leftrightarrow \frac{\cancel{A}}{\cancel{A}B_1}\vec{u}_\lambda=\lambda\vec{u}_\lambda$$
$$\frac{I_n}{B_1}\vec{u}_\lambda=\lambda\vec{u}_\lambda.$$

Theorem 2.7. If a matrix $\frac{A}{B}$ has an eigenvalue λ and an eigenvector $\vec{u}_\lambda$. Then there is a matrix A_1 such that eigenvalue λ and eigenvector $\vec{u}_\lambda$ are eigenvalue and eigenvector belong matrix A_1.

Proof. If $\frac{A}{B}\vec{u}_\lambda=\lambda\vec{u}_\lambda$ then

$$\frac{BA_1}{B}\vec{u}_\lambda = \lambda\vec{u}_\lambda \Leftrightarrow \frac{\cancel{B}A_1}{\cancel{B}}\vec{u}_\lambda = \lambda\vec{u}_\lambda$$
$$A_1\vec{u}_\lambda = \lambda\vec{u}_\lambda .$$

Theorem 2.8. Let $AB = C$. If $A\vec{u}_\lambda = \lambda\vec{u}_\lambda$ then,

$$\left(\frac{C^T}{B^T}\right)^T \vec{u}_\lambda = \lambda\vec{u}_\lambda .$$

3. RESULTS AND CONCLUSIONS

New approximations are given using some properties arising from the division of matrices. The study expresses that there are many properties of characteristic polynomials beyond the given lemmas and theorems.

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