Kuvvet Serileri

$\{a_n\}$, $a \in \mathbb{R}$ ve $x \in \mathbb{R}$ değişken olmak üzere

$$\sum_{n=0}^{+\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n + \dots$$

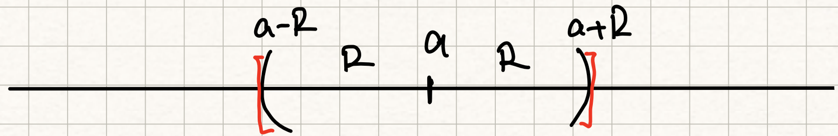
ifadesine bir kuvvet serisi denir.

$x=3 \Rightarrow \sum_{n=0}^{+\infty} a_n(3-a)^n$ reel sayı serisidir.

$x=-\pi \Rightarrow \sum_{n=0}^{+\infty} a_n(-\pi-a)^n$ " " " .

$x=a \Rightarrow \sum_{n=0}^{+\infty} a_n(a-a)^n = \sum_{n=0}^{+\infty} a_n 0^n = \sum_{n=0}^{+\infty} 0$ reel sayı serisidir.

$\sum_{n=0}^{+\infty} a_n(x-a)^n$ kuvvet serisi



Yakınsaklık merkezi $=a$

" yarıçapı $=R$

" aralığı $= (a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R]$

$R=+\infty \Rightarrow$ Y.A $= (-\infty, +\infty)$ olur

$R=0 \Rightarrow$ Y.A $= \{a\}$

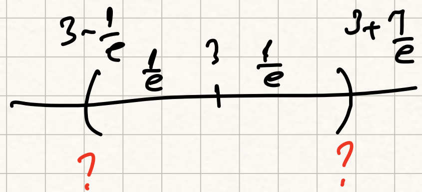
Örnekler:

1) $\displaystyle\sum_{n=1}^{+\infty} \frac{e^n}{\ln(n+1)}(x-3)^n$, Y.M, Y.Y, Y.A $=?$

$$\lim_{n\to+\infty}\left|\frac{\frac{\cancel{e^{n+1}}}{\ln(n+2)}(x-3)^{\cancel{n}+1}}{\frac{\cancel{e^{n}}}{\ln(n+1)}\cancel{(x-3)^{n}}}\right| = \lim_{n\to+\infty}\left|\frac{\ln(n+1)}{\ln(n+2)}\, e\,(x-3)\right|$$

$$=\lim_{n\to+\infty}\underbrace{\frac{\ln(n+1)}{\ln(n+2)}}_{1}\ \underbrace{e}_{e}\ \underbrace{|x-3|}_{|x-3|} = 1.e.|x-3| < 1$$

$$\Rightarrow |x-3| < \frac{1}{e}$$



$$\Rightarrow -\frac{1}{e} < x-3 < \frac{1}{e}$$

$$\Rightarrow 3-\frac{1}{e} < x < 3+\frac{1}{e}$$

$Y.M = 3$

$Y.Y = \frac{1}{e}$

$Y.A = \left[3-\frac{1}{e}, 3+\frac{1}{e}\right)$

$\left(-\infty, 3-\frac{1}{e}\right) \cup \left(3+\frac{1}{e}, +\infty\right)$ ıraksaktır.

$$x = 3-\frac{1}{e} \Rightarrow \sum_{n=1}^{+\infty}\frac{e^n}{\ln(n+1)}\left(\cancel{3}-\frac{1}{e}-\cancel{3}\right)^n = \sum_{n=1}^{+\infty}\frac{\cancel{e^n}}{\ln(n+1)}\frac{(-1)^n}{\cancel{e^n}}$$

$$=\sum_{n=1}^{+\infty}(-1)^n\frac{1}{\ln(n+1)}$$

$0 < \frac{1}{\ln(n+2)} \leq \frac{1}{\ln(n+1)}$ ve $\lim_{n\to+\infty}\frac{1}{\ln(n+1)} = 0$ Leibnitz Testinden, yakınsaktır.

$x=3+\frac{1}{e} \Rightarrow \sum_{n=1}^{+\infty} \frac{e^n}{\ln(n+1)}\left(\cancel{3}+\frac{1}{e}-\cancel{3}\right)^n = \sum_{n=1}^{+\infty} \frac{\cancel{e^n}}{\ln(n+1)} \frac{1}{\cancel{e^n}}$

$$=\sum_{n=1}^{+\infty} \frac{1}{\ln(n+1)}$$

$\forall x \in (0,+\infty)$ için $\ln x < x$ 'dir.

" " $\frac{1}{x} < \frac{1}{\ln x}$

Her $n \in \mathbb{N}$ için $\frac{1}{n+1} < \frac{1}{\ln(n+1)}$

$$\Rightarrow \frac{1}{2n} = \frac{1}{n+n} \leq \frac{1}{n+1} < \frac{1}{\ln(n+1)}$$

$\sum_{n=1}^{+\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{1}{n} \Rightarrow$ Karş. Testi $\sum_{n=1}^{+\infty} \frac{1}{\ln(n+1)}$ ıraksak.

$\Downarrow$

p testi ıraksak.

2) $\sum_{n=0}^{+\infty} \frac{(6-x)^{n+1}}{\sqrt{2n+1}}$ Y.M, Y.Y., Y.A =?

$$\lim_{n\to+\infty} \left| \frac{\frac{(6-x)^{\cancel{n+2}\,1}}{\sqrt{2n+3}}}{\frac{\cancel{(6-x)^{n+1}}}{\sqrt{2n+1}}} \right| = \lim_{n\to+\infty} \left| \frac{\sqrt{2n+1}}{\sqrt{2n+3}} (6-x) \right|$$

$$= \lim_{n\to+\infty} \underbrace{\frac{\sqrt{2n+1}}{\sqrt{2n+3}}}_{1} |6-x| = |6-x| < 1$$

$$\Rightarrow -1 < 6-x < 1$$

Y.M $= 6$

Y.Y $= 1$ $\qquad \Rightarrow -1 < x-6 < 1$

Y.A $= (5,7)$ $\qquad \Rightarrow 5 < x < 7$

3) $\displaystyle\sum_{n=1}^{+\infty} \frac{5^n}{(2n)!} x^{2n}$, Y.M. , Y.Y. , Y.A = ?

$$\lim_{n\to+\infty} \left| \frac{\dfrac{5^{\cancel{n}+1}}{(2n+2)!} x^{\cancel{2n}+2}}{\dfrac{\cancel{5^n}}{(2n)!} \cancel{x^{2n}}} \right| = \lim_{n\to+\infty} \left| \frac{\cancel{(2n)!}}{\underbrace{\cancel{(2n+2)!}}_{(2n+1)(2n+2)}} 5 \, x^2 \right|$$

$$= \lim_{n\to+\infty} \underbrace{\frac{5}{(2n+1)(2n+2)}}_{0} |x^2| = 0 \, |x^2| = 0 < 1$$

Y.M $= 0$

Y.Y $= +\infty$

Y.A $= (-\infty, +\infty)$

4) $\displaystyle\sum_{n=2}^{+\infty} \frac{n!}{n \ln n . 8^n} (x+1)^n$, Y.M, Y.Y, Y.A = ?

$$\lim_{n\to+\infty}\left|\frac{\dfrac{(n+1)!}{(n+1)\ln(n+1)\,3^{\cancel{n}+1}}\,(x+1)^{\cancel{n}+1}}{\dfrac{n!}{n\ln n\cdot\cancel{3^{n}}}\,\cancel{(x+1)^{n}}}\right| = \lim_{n\to+\infty}\left|\frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{n!}}\;\frac{n}{\cancel{n+1}}\;\frac{\ln n}{\ln(n+1)}\;\frac{1}{3}\,(x+1)\right|$$

$$=\lim_{n\to+\infty}\frac{n}{3}\;\frac{\ln n}{\ln(n+1)}\;|x+1| = \begin{cases} 0, & x=-1 \\ +\infty, & x\neq -1 \end{cases}$$

$Y.M = -1$

$Y.Y = 0$

$Y.A = \{-1\}$

5) $\displaystyle\sum_{n=1}^{+\infty}\frac{n!}{n^n}(x-5)^{\frac{n}{3}}$, $Y.M$, $Y.Y$, $Y.A = ?$

$$\lim_{n\to+\infty}\left|\frac{\dfrac{(n+1)!}{(n+1)^{n+1}}\,(x-5)^{\frac{\cancel{n}+1}{3}}}{\dfrac{n!}{n^n}\,(x-5)^{\cancel{\frac{n}{3}}}}\right| = \lim_{n\to+\infty}\left|\frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{n!}}\;\frac{n^n}{(n+1)^{n+\cancel{1}}}\,(x-5)^{1/3}\right|$$

$$=\lim_{n\to+\infty}\underbrace{\frac{n^n}{(n+1)^n}}_{e^{-1}}\,|x-5|^{1/3} = \frac{1}{e}\cdot|x-5|^{1/3} < 1$$

$\Rightarrow |x-5|^{1/3} < e$

$\Rightarrow |x-5| < e^3$

$-e^3 < x-5 < e^3$

$$\Rightarrow 5-e^3 < x < 5+e^3$$

Y.M $=5$

Y.Y $= e^3$

Y.A $= (5-e^3, 5+e^3)$

$$\lim_{n\to+\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^n \overset{1^\infty}{=} e^{-1}$$

$$\lim_{n\to+\infty} n \ln\left(\frac{n}{n+1}\right) \overset{0.\infty}{=} \lim_{n\to+\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} \overset{\frac{0}{0}}{=} \lim_{n\to+\infty} \frac{\frac{\frac{1.(n+1)-1.n}{(n+1)^{\cancel{2}}}}{\frac{\cancel{n}}{\cancel{n+1}}}}{-\frac{1}{n^{\cancel{2}}}}$$

$$= \lim_{n\to+\infty} \frac{\frac{1}{n+1}}{-\frac{1}{n}} = \lim_{n\to+\infty} \frac{-n}{n+1} = -1$$

Alıştırmalar:

Aşağıdaki kuvvet serilerinin Y.M., Y.Y., Y.A. ?

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} x^n, \quad \sum_{n=1}^{+\infty} \frac{5^n}{n!}(x+2)^{n+2}, \quad \sum_{n=1}^{+\infty} \frac{(4x-5)^n}{3^n}$$

$$\sum_{n=2}^{+\infty} \frac{(-1)^n(x-1)^n}{n\ln n}, \quad \sum_{n=1}^{+\infty} \frac{3^n}{(-2)^n n(n+1)}(x+5)^n, \quad \sum_{n=1}^{+\infty} \frac{n^2}{3^{2n}}(x+7)^{3n}$$

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n}(x-2)^{\frac{n}{3}}, \quad \sum_{n=0}^{+\infty} \frac{5^n}{(2n)!}\left(\frac{x+1}{3}\right)^n, \quad \sum_{n=1}^{+\infty} \left(\frac{n}{n+1}\right)^{n^2}(x-e)^n$$

$$\sum_{n=0}^{+\infty} \left(\frac{x^2+2}{6}\right)^{n^2}$$