Taylor Serileri:

$f: A \subset \mathbb{R} \to \mathbb{R}$ her mertebeden türevlenebilir bir fonksiyon $a \in A$ olsun.

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + $$
$$+ \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

kuvvet serisine f fonksiyonunun a'daki taylor serisi denir. ($a=0$ için Maclaurin serisi)

Örnekler:

$f(x) = \dfrac{a}{bx+c}$

$f(x) = a\ln(bx+c)$

$f(x) = a\, e^{bx+c}$

$f(x) = a\sqrt{bx+c}$

1) $f(x) = \sqrt[3]{\dfrac{4}{e^{2x+3}}}$, $a=2$ Taylor?

$$f(x) = \sqrt[3]{\frac{4}{e^{2x+3}}} = \frac{\sqrt[3]{4}}{\sqrt[3]{e^{2x+3}}} = \frac{\sqrt[3]{4}}{(e^{2x+3})^{1/3}}$$

$$= \frac{\sqrt[3]{4}}{e^{\frac{2x+3}{3}}} = \frac{\sqrt[3]{4}}{e^{\frac{2}{3}x+1}} = \sqrt[3]{4}\; e^{-\frac{2}{3}x-1}$$

$$f(x) = \sqrt[3]{4}\; e^{-\frac{2}{3}x-1}$$

$$f^{(1)}(x) = \sqrt[3]{4} \cdot \left(-\frac{2}{3}\right) \cdot e^{-\frac{2}{3}x-1}$$

$$f^{(2)}(x) = \sqrt[3]{4} \cdot \left(-\frac{2}{3}\right) \cdot \left(-\frac{2}{3}\right) \cdot e^{-\frac{2}{3}x-1}$$

$$f^{(3)}(x) = \sqrt[3]{4} \cdot \left(-\frac{2}{3}\right) \cdot \left(-\frac{2}{3}\right) \cdot \left(-\frac{2}{3}\right) e^{-\frac{2}{3}x-1}$$

$$\vdots$$

$$f^{(n)}(x) = \sqrt[3]{4}\left(-\frac{2}{3}\right)^n \cdot e^{-\frac{2}{3}x-1} \quad , n = 0, 1, 2, 3, \ldots$$

$$f^{(n)}(2) = \sqrt[3]{4}\left(-\frac{2}{3}\right)^n \quad e^{-\frac{7}{3}}$$

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{+\infty} \frac{\sqrt[3]{4}\left(-\frac{2}{3}\right)^n \; e^{-\frac{7}{3}}}{n!}(x-2)^n$$

2) $f(x) = \dfrac{3}{5x+1}$, $a = -1$ Taylor ?

$$f(x) = \frac{3}{5x+1} = 3.(5x+1)^{-1}$$

$$f^{(1)}(x) = 3 \,.\, 5.(-1)\,(5x+1)^{-2}$$

$$f^{(2)}(x) = 3.\,5.\,(-1)\,.5.(-2)\,(5x+1)^{-3}$$

$$f^{(3)}(x) = 3.\,5.(-1).\,5.\,(-2).\,5.\,(-3)\,(5x+1)^{-4}$$

$$\vdots$$

$$f^{(n)}(x) = 3 \,.\, 5^n . (-1)^n . n!\,(5x+1)^{-(n+1)} \quad , n = 0, 1, 2, 3, \ldots$$

$$f^{(n)}(-1) = 3.\,5^n\,(-1)^n .\, n!\,(-4)^{-(n+1)}$$

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n = \sum_{n=0}^{+\infty} \frac{3.5^n(-1)^n\,\cancel{n!}\,(-4)^{-(n+1)}}{\cancel{n!}}(x+1)^n$$

3) $f(x) = 3\ln\left(\sqrt[4]{\frac{6}{4x+7}}\right)$, $x_0 = 2$ Taylor, $x=0$ Taylor Maclaurin?

$$f(x) = 3\ln\left(\sqrt[4]{\frac{6}{4x+7}}\right) = 3\ln\left(\sqrt[4]{\frac{1}{\frac{4}{6}x+\frac{7}{6}}}\right)$$

$$= 3\ln\left(\frac{1}{\left(\frac{4}{6}x+\frac{7}{6}\right)^{1/4}}\right) = 3\ln\left(\left(\frac{4}{6}x+\frac{7}{6}\right)^{-1/4}\right)$$

$$f(x) = \frac{-3}{4}\ln\left(\frac{2}{3}x+\frac{7}{6}\right)$$

$$f^{(1)}(x) = \frac{-3}{4}\cdot\frac{2}{3}\cdot\frac{1}{\frac{2}{3}x+\frac{7}{6}} = -\frac{\cancel{3}}{4}\cdot\frac{\cancel{2}}{3}\cdot\left(\frac{2}{3}x+\frac{7}{6}\right)^{-1}$$

$$f^{(2)}(x) = -\frac{\cancel{3}}{4}\cdot\frac{\cancel{2}}{3}\cdot\frac{\cancel{2}}{3}\cdot(-1)\left(\frac{2}{3}x+\frac{7}{6}\right)^{-2}$$

$$f^{(3)}(x) = -\frac{\cancel{3}}{4}\cdot\frac{\cancel{2}}{3}\cdot\frac{\cancel{2}}{3}\cdot(-1)\cdot\frac{\cancel{2}}{3}\cdot(-2)\cdot\left(\frac{2}{3}x+\frac{7}{6}\right)^{-3}$$

$$f^{(4)}(x) = -\frac{\cancel{3}}{4}\cdot\frac{\cancel{2}}{3}\cdot\frac{\cancel{2}}{3}\cdot(-1)\cdot\frac{\cancel{2}}{3}\cdot(-2)\cdot\frac{\cancel{2}}{3}\cdot(-3)\left(\frac{2}{3}x+\frac{7}{6}\right)^{-4}$$

$$\vdots$$

$$f^{(n)}(x) = -\frac{3}{4}\left(\frac{2}{3}\right)^n(-1)^{n-1}(n-1)!\cdot\left(\frac{2}{3}x+\frac{7}{6}\right)^{-n}, \quad n=1,2,3,\ldots$$

$$f^{(n)}(2) = -\frac{3}{4}\left(\frac{2}{3}\right)^n(-1)^{n-1}(n-1)!\left(\frac{5}{2}\right)^{-n}$$

$$\sum_{n=0}^{+\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n = f(2) + \sum_{n=1}^{+\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$= -\frac{3}{4}\ln\left(\frac{5}{2}\right) + \sum_{n=1}^{+\infty} \frac{-\frac{3}{4}\left(\frac{2}{3}\right)^n (-1)^{n-1} (n-1)! \left(\frac{5}{2}\right)^{-n}}{n!} (x-2)^n$$

$$f^{(n)}(0) = \frac{-3}{4}\left(\frac{2}{3}\right)^n (-1)^{n-1} (n-1)! \left(\frac{7}{6}\right)^{-n}$$

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= -\frac{3}{4}\ln\left(\frac{7}{6}\right) + \sum_{n=1}^{+\infty} \frac{-\frac{3}{4}\left(\frac{2}{3}\right)^n (-1)^{n-1} (n-1)! \left(\frac{7}{6}\right)^n}{n!} x^n$$

4) $f(x) = 3\sqrt{4x-3}$, $x=1$ Taylor ?

$$f(x) = 3 \cdot (4x-3)^{1/2}$$

$$f^{(1)}(x) = 3 \cdot 4 \cdot \frac{1}{2} (4x-3)^{-1/2}$$

$$f^{(2)}(x) = 3 \cdot 4 \cdot \frac{1}{2} \cdot 4 \cdot \left(-\frac{1}{2}\right) (4x-3)^{-\frac{3}{2}}$$

$$f^{(3)}(x) = 3 \cdot 4 \cdot \frac{1}{2} \cdot 4 \cdot \left(-\frac{1}{2}\right) \cdot 4 \cdot \left(-\frac{3}{2}\right) (4x-3)^{-\frac{5}{2}}$$

$$f^{(4)}(x) = 3 \cdot 4 \cdot \frac{1}{2} \cdot 4 \cdot \left(-\frac{1}{2}\right) \cdot 4 \cdot \left(-\frac{3}{2}\right) \cdot 4 \cdot \left(\frac{-5}{2}\right) (4x-3)^{-\frac{7}{2}}$$

$\vdots$

$$f^{(n)}(x) = 3 \cdot 4^n \frac{1}{2^n} (-1)^{n-1}\, 1.3.5 \cdots (2n-3)\, (4x-3)^{-\frac{(2n-1)}{2}}$$

$n = 2, 3, 4, \ldots$

$$f^{(n)}(1) = 3 \cdot 4^n \frac{1}{2^n} (-1)^n\, 1.3.5 \cdots (2n-3)$$

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = f(1) + f^{(1)}(1)(x-1) + \sum_{n=2}^{+\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= 3 + 6.(x-1) + \sum_{n=2}^{+\infty} \frac{3.4^n.\frac{1}{2^n}(-1)^n \, 1.3.5...(2n-3)}{n!}(x-1)^n$$

Alıştırmalar:

1) $f(x)=\frac{3}{x+3}$, $x_0=2$ Taylor?

2) $f(x)=\sqrt{e^{3x-1}}$, $x_0=4$ " ? Maclaurin?

3) $f(x)=3\ln(4x-12)$, $x_0=5$ Taylor?

4) $f(x)=\sqrt{3-2x}$, $x_0=1$ Taylor?

5) $f(x)=\frac{4x+1}{(3x-7)(2x+1)}$, $x_0=2$ Taylor?

6) $f(x)=\frac{3}{(2x+3)^3}$, $x_0=1$ Taylor?