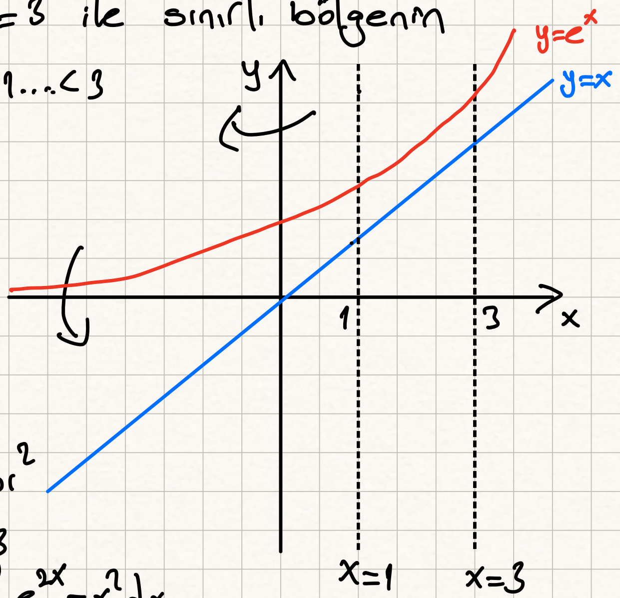
Örnek: $y=e^x$, $y=x$, $x=1$, $x=3$ ile sınırlı bölgenin

a) Alanı ? $e=2{,}71\ldots<3$

b) x ekseni hacim ?

c) y ekseni hacim ?



a) $$A=\int_1^3 e^x-x\,dx = e^x-\frac{x^2}{2}\Big|_1^3$$

$$= e^3-\frac{9}{2}-\left(e-\frac{1}{2}\right) = e^3-e-4 \text{ br}^2$$

b) $$V_x=\pi\int_1^3 (e^x)^2-x^2\,dx = \pi\int_1^3 e^{2x}-x^2\,dx$$

$$=\pi\left(\frac{e^{2x}}{2}-\frac{x^3}{3}\Big|_1^3\right) = \pi\left[\left(\frac{e^6}{2}-9\right)-\left(\frac{e^2}{2}-\frac{1}{3}\right)\right]$$

$$=\pi\left(\frac{e^6-e^2}{2}-\frac{26}{3}\right) \text{ br}^3$$

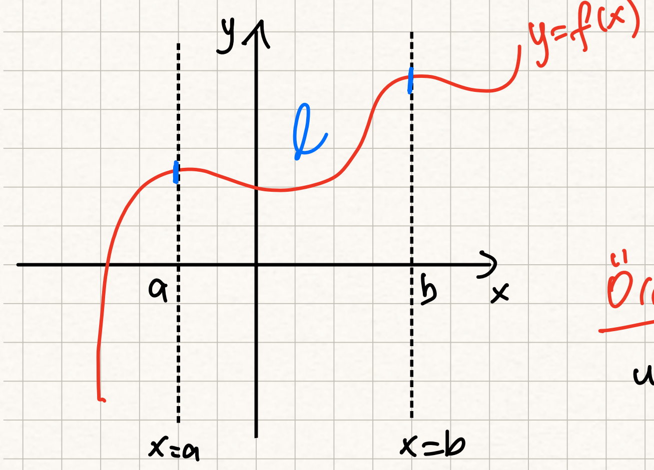
c) $$V_y=2\pi\int_1^3 x(e^x-x)\,dx = 2\pi\int_1^3 xe^x-x^2\,dx = 2\pi\left(xe^x-e^x-\frac{x^3}{3}\Big|_1^3\right)$$

$$=2\pi\left(\left(3e^3-e^3-9\right)-\left(\cancel{e^1}-\cancel{e^1}-\frac{1}{3}\right)\right) = 2\pi\left(2e^3-\frac{26}{3}\right) \text{ br}^3$$

$$\int xe^x\,dx = xe^x-\int e^x\,dx = xe^x-e^x+c$$

$$\boxed{\begin{array}{ll} x=u & e^x dx = dv \\ dx=du & e^x = v \end{array}}$$

3) Eğri uzunluğu:



$y=f(x)$ eğrisinin $x=a$, $x=b$ ile sınırlı parçasının uzunluğu

$$l=\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Örnek: $f(x)=\sqrt{2x-x^2}$, $\frac{1}{2}\le x\le\frac{3}{2}$ uzunluğu?

$$l=\int_{1/2}^{3/2}\sqrt{1+\left(\left(\sqrt{2x-x^2}\right)'\right)^2}\,dx=\int_{1/2}^{3/2}\sqrt{1+\left((2-2x)\cdot\frac{1}{2\sqrt{2x-x^2}}\right)^2}\,dx$$

$$=\int_{1/2}^{3/2}\sqrt{1+\frac{1-2x+x^2}{2x-x^2}}\,dx=\int_{1/2}^{3/2}\sqrt{\frac{\cancel{2x}-\cancel{x^2}+1-\cancel{2x}+\cancel{x^2}}{2x-x^2}}\,dx$$

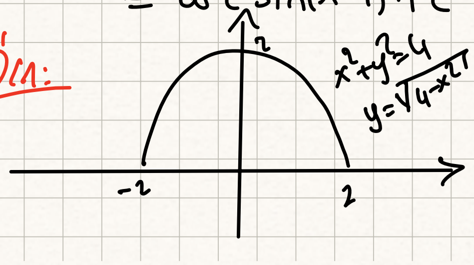
$$=\int_{1/2}^{3/2}\frac{1}{\sqrt{2x-x^2}}\,dx=\arcsin(x-1)\Big|_{1/2}^{3/2}=\arcsin\frac{1}{2}-\arcsin\left(-\frac{1}{2}\right)$$

$$=\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)=\frac{\pi}{3}\text{ br.}$$

$$\int\frac{1}{\sqrt{2x-x^2}}\,dx=\int\frac{1}{\sqrt{-(x^2-2x+1)+1}}\,dx=\int\frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

$$=\arcsin(x-1)+c$$

Örn:



$$l=\frac{2\pi.2}{2}=2\pi\text{ br}$$

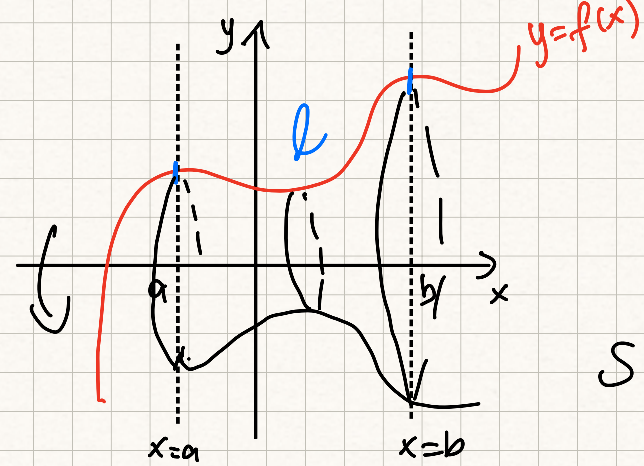
$$l=\int_{-2}^{2}\sqrt{1+\left(\left(\sqrt{4-x^2}\right)'\right)^2}\,dx$$

$$= \int_{-2}^{2} \sqrt{1+\left(\frac{-2x}{2\sqrt{4-x^2}}\right)^2}\,dx = \int_{-2}^{2}\sqrt{1+\frac{x^2}{4-x^2}}\,dx = \int_{-2}^{2}\sqrt{\frac{4}{4-x^2}} = \int_{-2}^{2}\sqrt{\frac{1}{\frac{4}{4}-\frac{x^2}{4}}}\,dx$$

$$= \int_{-2}^{2} \frac{1}{\sqrt{1-(\frac{x}{2})^2}}\,dx = 2\arcsin\left(\frac{x}{2}\right)\Big|_{-2}^{2} = 2\arcsin 1 - 2\arcsin(-1)$$

$$= \underbrace{2\,\frac{\pi}{2}}_{\pi} - \underbrace{2\left(-\frac{\pi}{2}\right)}_{\pi} = 2\pi \text{ br}$$

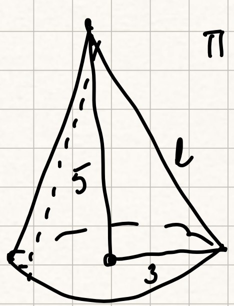
4) Yüzey Alanı



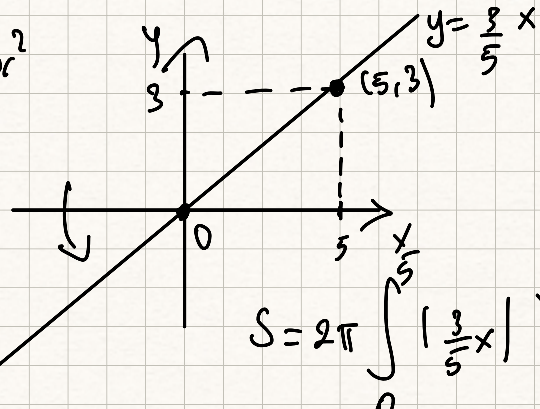
$y=f(x)$ eğrisinin $a \le x \le b$ arasındaki parçası x ekseni etrafında dönerse oluşacak dönel cismin yüzey alanı

$$S = 2\pi \int_a^b |f(x)| \sqrt{1+(f'(x))^2}\,dx$$

Örnek: Şekildeki dik dairesel koninin yüzey alanı?



$\pi\, 3.\sqrt{34}$ br^2



$$S = 2\pi \int_0^5 \left|\frac{3}{5}x\right| \sqrt{1+\left(\left(\frac{3}{5}x\right)'\right)^2}\,dx$$

$$= 2\pi \int_0^5 \frac{3}{5}x \sqrt{1+\frac{9}{25}}\,dx = 2\pi\,\frac{3}{5}.\frac{\sqrt{34}}{5}\int_0^5 x\,dx = \frac{6\sqrt{34}\,\pi}{25}\left(\frac{x^2}{2}\Big|_0^5\right)$$

$$= \frac{\cancel{6}\sqrt{34}\,\pi}{\cancel{25}}.\frac{\cancel{25}}{\cancel{2}} = 3\sqrt{34}\,\pi$$

Alıştırmalar:

Aşağıdaki eğri ve doğrularla sınırlı bölgenin alanını, x ve y ekseni etrafında döndürülmesi ile oluşacak dönel cisimlerin hacimlerini hesaplayınız.

1) $y=\frac{1}{\cos x}$, $y=\tan x$, $x=0$, $x=1$

2) $y=\sqrt{1-x^2}$, $y=0$

3) $y=x^2+1$, $y=x+3$

4) $y=x^3$, $x=-1$, $x=1$

5) $y=3x^2-x^3$, $y=0$

Aşağıda verilen eğrilerin uzunluklarını ve x ekseni etrafında dönmesi ile oluşacak dönel cisimlerin yüzey alanlarını hesaplayınız.

6) $y=2\sqrt{x}$, $1\leq x\leq 2$

7) $y=1-x$, $0\leq x\leq 1$

8) $y=\frac{x^3}{9}$, $0\leq x\leq 2$

9) $y=2\sqrt{4-x}$, $0\leq x\leq \frac{15}{4}$

10) $y=\frac{x^{3/2}}{3}-x^{1/2}$, $1\leq x\leq 3$

11) $y=\sqrt{\frac{x^2-x^4}{4}}$, $-1\leq x\leq 1$