Taylor Serileri

$f: A \subset \mathbb{R} \to \mathbb{R}$ fonksiyonu bir $a \in A$'da her mertebeden türevlenebilir olsun.

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

kuvvet serisine $f(x)$ fonksiyonunun a noktasındaki Taylor serisi denir. (0'daki Taylor serisi = Maclaurin serisi)

Örnekler:

$f(x) = \frac{a}{bx+c}$

$f(x) = a\ln(bx+c)$

$f(x) = a\sqrt{bx+c}$

$f(x) = a\, e^{bx+c}$

1) $f(x) = \sqrt[3]{\frac{5}{e^{2x+7}}}$, $a=1$ Taylor?

$$f(x) = \sqrt[3]{\frac{5}{e^{2x+7}}} = \frac{\sqrt[3]{5}}{\sqrt[3]{e^{2x+7}}} = \frac{\sqrt[3]{5}}{e^{\frac{2}{3}x+\frac{7}{3}}}$$

$$= \sqrt[3]{5}\, e^{-\frac{2}{3}x-\frac{7}{3}}$$

$$f^{(1)}(x) = \sqrt[3]{5} \cdot \left(-\frac{2}{3}\right) \cdot e^{-\frac{2}{3}x-\frac{7}{3}}$$

$$f^{(2)}(x) = \sqrt[3]{5} \cdot \left(-\frac{2}{3}\right) \cdot \left(-\frac{2}{3}\right) \cdot e^{-\frac{2}{3}x-\frac{7}{3}}$$

$$f^{(3)}(x) = \sqrt[3]{5}\cdot\left(-\frac{2}{3}\right)\cdot\left(-\frac{2}{3}\right)\cdot\left(-\frac{2}{3}\right) e^{-\frac{2}{3}x-\frac{7}{3}}$$

$$\vdots$$

$$f^{(n)}(x) = \sqrt[3]{5}\left(-\frac{2}{3}\right)^n \cdot e^{-\frac{2}{3}x-\frac{7}{3}} \quad , \; n = 0,1,2,3,\dots$$

$$f^{(n)}(1) = \sqrt[3]{5}\left(-\frac{2}{3}\right)^n e^{-3}$$

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{+\infty} \frac{\sqrt[3]{5}\left(-\frac{2}{3}\right)^n e^{-3}}{n!}(x-1)^n$$

2) $f(x) = \dfrac{-2}{4x-1}$, $a = -2$ Taylor ?

$$f(x) = -2\cdot(4x-1)^{-1}$$

$$f^{(1)}(x) = -2\cdot 4\cdot(-1)\cdot(4x-1)^{-2}$$

$$f^{(2)}(x) = -2\cdot 4\cdot(-1)\cdot 4\cdot(-2)\,(4x-1)^{-3}$$

$$f^{(3)}(x) = -2\cdot 4\cdot(-1)\cdot 4\cdot(-2)\cdot 4(-3)\cdot(4x-1)^{-4}$$

$$\vdots$$

$$f^{(n)}(x) = -2\cdot 4^n(-1)^n\cdot n!\,(4x-1)^{-(n+1)} \quad , \; n = 0,1,2,3,\dots$$

$$f^{(n)}(-2) = -2\cdot 4^n\cdot(-1)^n\cdot n!\,(-9)^{-(n+1)}$$

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(-2)}{n!}(x+2)^n = \sum_{n=0}^{+\infty} \frac{-2\cdot 4^n(-1)^n\, n!\,(-9)^{-(n+1)}}{n!}(x+2)^n$$

3) $f(x)=\ln\left(\sqrt{\dfrac{5}{\sqrt[3]{2-3x}}}\right)$, Maclaurin serisi?

$$f(x)=\ln\left(\sqrt{\sqrt[3]{\frac{125}{2-3x}}}\right)=\ln\left(\sqrt[6]{\frac{1}{\frac{2}{125}-\frac{3}{125}x}}\right)$$

$$=\ln\left[\left(\left(\frac{2}{125}-\frac{3}{125}x\right)^{-1}\right)^{1/6}\right]=\ln\left(\left(\frac{2}{125}-\frac{3}{125}x\right)^{-1/6}\right)$$

$$=-\frac{1}{6}\ln\left(-\frac{3}{125}x+\frac{2}{125}\right)$$

$$f^{(1)}(x)=-\frac{1}{6}\cdot-\frac{3}{125}\cdot\frac{1}{-\frac{3}{125}x+\frac{2}{125}}=-\frac{1}{6}\cdot-\frac{3}{125}\left(-\frac{3}{125}x+\frac{2}{125}\right)^{-1}$$

$$f^{(2)}(x)=-\frac{1}{6}\cdot\frac{-3}{125}\;\frac{-3}{125}\;(-1)\;\left(-\frac{3}{125}x+\frac{2}{125}\right)^{-2}$$

$$f^{(3)}(x)=-\frac{1}{6}\cdot\frac{-3}{125}\cdot\frac{-3}{125}\cdot(-1)\cdot\frac{-3}{125}\cdot(-2)\cdot\left(-\frac{3}{125}x+\frac{2}{125}\right)^{-3}$$

$$f^{(4)}(x)=-\frac{1}{6}\cdot\frac{-3}{125}\cdot\frac{-3}{125}\cdot(-1)\cdot\frac{-3}{125}\cdot(-2)\cdot\frac{-3}{125}\cdot(-3)\cdot\left(-\frac{3}{125}x+\frac{2}{125}\right)^{-4}$$

$\vdots$

$$f^{(n)}(x)=-\frac{1}{6}\left(\frac{-3}{125}\right)^{n}(-1)^{n-1}(n-1)!\,\left(-\frac{3}{125}x+\frac{2}{125}\right)^{-n}$$

$n=1,2,3,\ldots$

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= -\frac{1}{6}\ln\left(\frac{2}{125}\right) + \sum_{n=1}^{+\infty} \frac{-\frac{1}{6}\left(\frac{-3}{125}\right)^n \cdot (-1)^{n-1} (n-1)! \left(\frac{2}{125}\right)^{-n}}{n!} x^n$$

4) $f(x) = 3\sqrt{5x+2}$, $x=5$ Taylor serisi?

$f(x) = 3 \cdot (5x+2)^{1/2}$ $\qquad 1.3.5 \ldots (-3)$

$f^{(1)}(x) = 3 \cdot 5 \cdot \frac{1}{2} \cdot (5x+2)^{-1/2}$

$f^{(2)}(x) = 3 \cdot 5 \cdot \frac{1}{2} \cdot 5 \cdot \left(-\frac{1}{2}\right)(5x+2)^{-\frac{3}{2}}$

$f^{(3)}(x) = 3 \cdot 5 \cdot \frac{1}{2} \cdot 5 \cdot \left(-\frac{1}{2}\right) \cdot 5 \cdot \left(-\frac{3}{2}\right)(5x+2)^{-\frac{5}{2}}$

$f^{(4)}(x) = 3 \cdot 5 \cdot \frac{1}{2} \cdot 5 \cdot \left(-\frac{1}{2}\right) \cdot 5 \cdot \left(-\frac{3}{2}\right) \cdot 5 \cdot \frac{-5}{2} \cdot (5x+2)^{-\frac{7}{2}}$

$\vdots$

$f^{(n)}(x) = 3 \cdot 5^n \frac{1}{2^n} (-1)^{n-1} \cdot 1.3.5 \ldots (2n-3)\,(5x+2)^{-\frac{(2n-1)}{2}}$

$n = 2, 3, 4, \ldots$

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n = f(5) + f^{(1)}(5)(x-5) + \sum_{n=2}^{+\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n$$

$$= 4\sqrt{3} + \frac{5}{2\sqrt{3}}(x-5) +$$

$$+ \sum_{n=2}^{+\infty} \frac{3 \cdot 5^n \cdot \frac{1}{2^n} (-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot 27^{-\frac{(2n-1)}{2}}}{n!} (x-5)^n$$

Alıştırmalar:

1) $f(x) = \frac{3}{x+3}$, $x_0 = 2$ Taylor?

2) $f(x) = \sqrt{e^{3x-1}}$, $x_0 = 4$ " ? Maclaurin?

3) $f(x) = 3\ln(4x-12)$, $x_0 = 5$ Taylor?

4) $f(x) = \sqrt{3-2x}$, $x_0 = 1$ Taylor?

5) $f(x) = \frac{4x+1}{(3x-7)(2x+1)}$, $x_0 = 2$ Taylor?

6) $f(x) = \frac{3}{(2x+3)^3}$, $x_0 = 1$ Taylor?