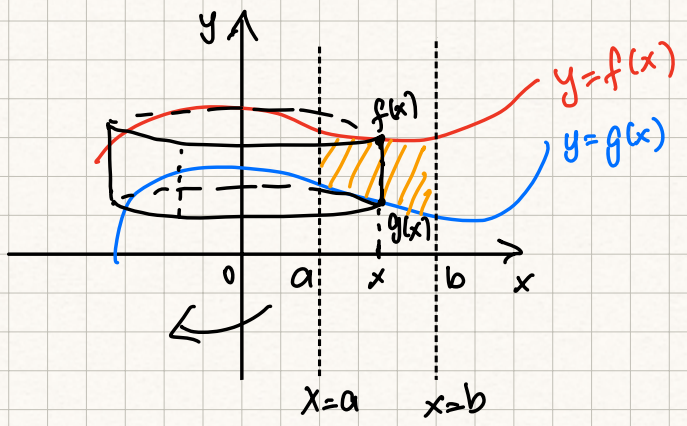
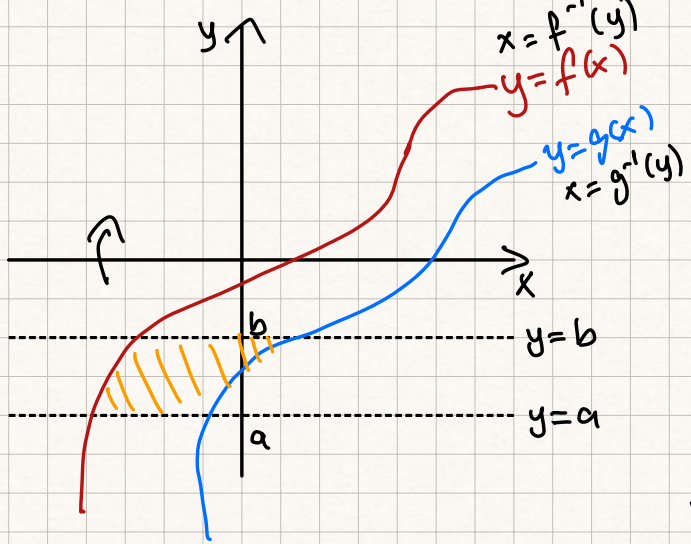
2. Kabuk Yöntemi



$y=f(x)$, $y=g(x)$, $x=a$, $x=b$ ile sınırlı bölge şekildeki gibi ise bu bölgenin y ekseni etrafında döndürülmesi ile oluşacak dönel cismin hacmi

$$V = 2\pi \int_a^b x\,(f(x)-g(x))\,dx$$ ile hesaplanır.

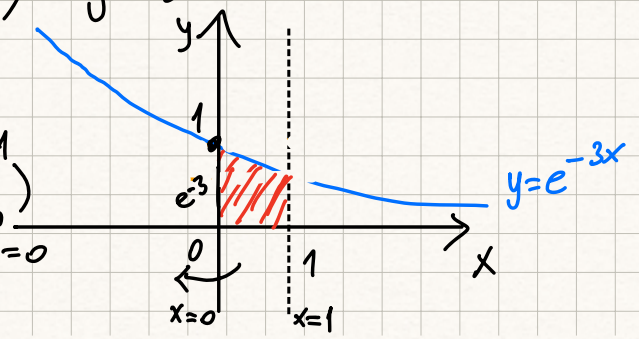


Bzş. $y=f(x)$, $y=g(x)$, $y=a$ $y=b$ ile sınırlı bölge şekildeki gibi ise, bu bölgenin x ekseni etrafında dönmesi ile oluşacak dönel cismin hacmi

$$V = 2\pi \int_a^b y\,(g^{-1}(y)-f^{-1}(y))\,dy$$

Örnekler:

1) $y=e^{-3x}$, $x=0$, $x=1$, $y=0$, y ekseni



$$V = 2\pi \int_0^1 x\,(e^{-3x}-0)\,dx$$

$$= 2\pi \int_0^1 x\,e^{-3x}\,dx = 2\pi \left(e^{-3x}\,\frac{3x-1}{-9} \Big|_0^1 \right)$$

$$= 2\pi \left(e^{-3}\,\frac{2}{-9} + \frac{1}{9} \right) br^3$$

$y=0$

$$\int x\, e^{-3x}\,dx = x\,\frac{e^{-3x}}{-3} - \int \frac{e^{-3x}}{-3}\,dx = x\,\frac{e^{-3x}}{-3} + \frac{1}{3}\int e^{-3x}\,dx$$

$x = u$, $e^{-3x}\,dx = dv$

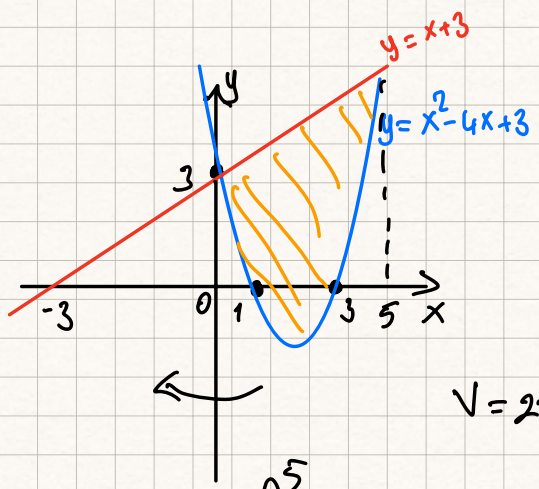
$dx = du$, $\frac{e^{-3x}}{-3} = v$

$$= x\,\frac{e^{-3x}}{-3} + \frac{1}{3}\,\frac{e^{-3x}}{-3} + C$$

$$= e^{-3x}\left(\frac{x}{-3} + \frac{1}{-9}\right) + C = e^{-3x}\left(\frac{3x+1}{-9}\right) + C$$

2) $y = x^2 - 4x + 3$, $y = x + 3$, y ekseni

$$y = x^2 - 4x + 3 = (x-1)(x-3)$$



$$x^2 - 4x + 3 = x + 3$$

$$x^2 - 5x = 0$$

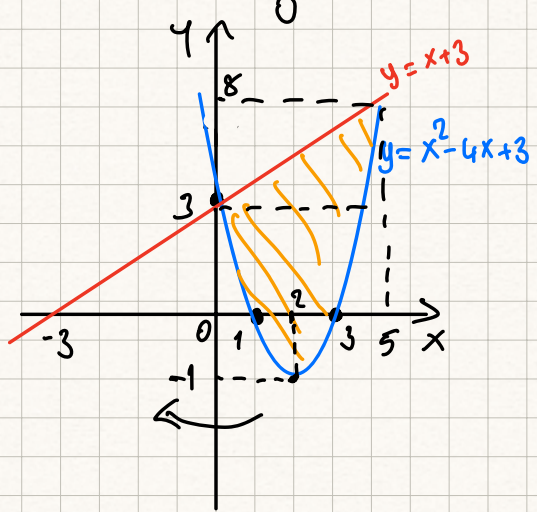
$$x(x-5) = 0$$

$$x = 0 \,,\; x = 5$$

$$V = 2\pi \int_0^5 x\left((x+3) - (x^2-4x+3)\right)dx$$

$$= 2\pi \int_0^5 -x^3 + 5x^2\,dx = 2\pi \left(-\frac{x^4}{4} + \frac{5x^3}{3}\Big|_0^5\right) = 2\pi\,\frac{-5^4}{4} + \frac{5^4}{3}$$

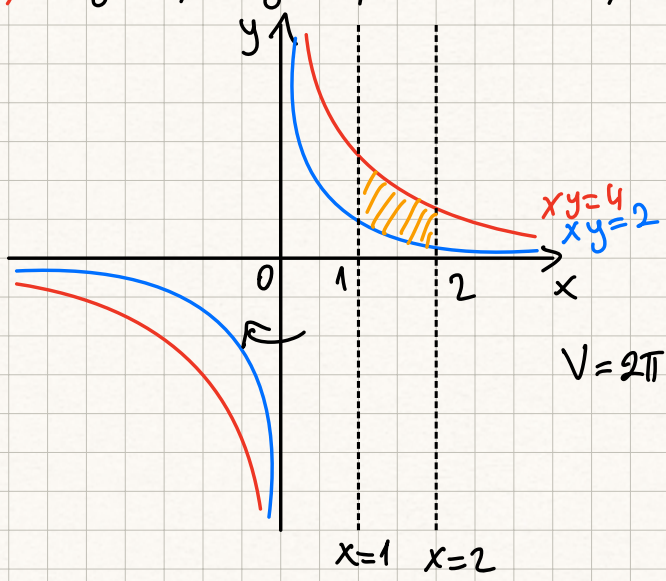
$$= 2\pi\,\frac{5^4}{12}\,br^3$$



$$V = \pi \int_{-1}^{3} \left(2 + \sqrt{y+1}\right)^2 - \left(2 - \sqrt{y+1}\right)^2 dy$$

$$+ \pi \int_3^8 \left(2 + \sqrt{y+1}\right)^2 - (y-3)^2\,dy = ?$$

3) $xy=2$, $xy=4$, $x=1$, $x=2$, y ekseni



$xy=2$

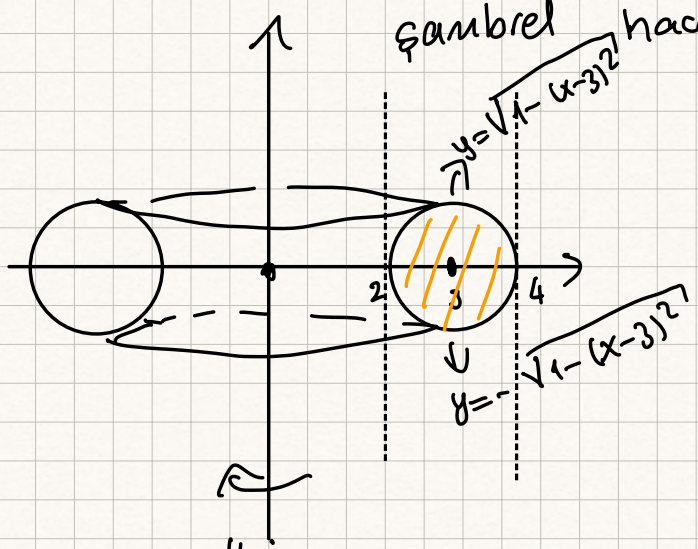
$y=\frac{2}{x}$

$y=2\cdot\frac{1}{x}$

$$V=2\pi\int_1^2 x\left(\frac{4}{x}-\frac{2}{x}\right)dx=4\pi\int_1^2 dx$$

$$=4\pi\ br^3$$

4) Şekildeki taralı bölgenin y ekseni dönmesi ile oluşan gambrel hacmi?



$x^2+y^2=1^2$

$(x-3)^2+y^2=1$

$y^2=1-(x-3)^2$

$y=\pm\sqrt{1-(x-3)^2}$

$$V=2\pi\int_2^4 x\left(\sqrt{1-(x-3)^2}-\left(-\sqrt{1-(x-3)^2}\right)\right)dx$$

$$=4\pi\int_2^4 x\sqrt{1-(x-3)^2}\,dx$$

$$\int x\sqrt{1-(x-3)^2}\,dx = \int (3+\sin t)\underbrace{\sqrt{1-\sin^2 t}}_{\cos t}\cos t\,dt$$

$x-3=\sin t$
$\arcsin(x-3)=t$
$dx=\cos t\,dt$

$\cos t = u$
$\sin t\,dt = -du$

$$= \int (3+\sin t)\cos^2 t\,dt$$

$$= \int 3\cos^2 t\,dt + \int \sin t\cos^2 t\,dt$$

$$= \int 3\,\frac{1+\cos 2t}{2}\,dt + \int -u^2\,du = \frac{3}{2}\int 1+\cos 2t\,dt + \left(-\frac{u^3}{3}\right)$$

$$= \frac{3}{2}\left(t+\frac{\sin 2t}{2}\right) - \frac{\cos^3 t}{3} + c$$

$$= \underbrace{\frac{3}{2}\left(\arcsin(x-3) + \frac{\sin 2(\arcsin(x-3))}{2}\right) - \frac{\cos^3(\arcsin(x-3))}{3} + c}_{f(x)}$$

$$V = 4\pi\left(f(x)\Big|_2^4\right)$$

$$= 4\pi\left[\frac{3}{2}\left(\arcsin 1 + \frac{\sin 2(\arcsin 1)}{2}\right) - \frac{\cos^3(\arcsin 1)}{3}\right)$$

$$- \left(\frac{3}{2}\left(\arcsin(-1) + \frac{\sin 2(\arcsin(-1))}{2}\right) - \frac{\cos^3(\arcsin(-1))}{3}\right)\Bigg]$$

$$= 4\pi\left[\left(\frac{3}{2}\left(\frac{\pi}{2}+0\right) - \frac{0^3}{3}\right) - \left(\frac{3}{2}\left(-\frac{\pi}{2}+0\right) - \frac{0^3}{3}\right)\right]$$

$$= 4\pi\left(\frac{3\pi}{4} + \frac{3\pi}{4}\right) = 6\pi^2\ bc^3$$