KTÜ/Matematik Bölümü **Öğrencinin; İmza:** **Süre:110dk**

MAT 2010-Lineer Cebir-II/2023-2024 Bahar Dönemi **Adı-Soyadı:**

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SORULAR

(15p) **S.1)** $f: M_2(\mathbb{R}) \to \mathbb{R}^3$, $f\left(\begin{bmatrix} x & y \\ z & t \end{bmatrix}\right) = (x+y, y+z, z+t)$ lineer dönüşümü verilsin. $T = \left\{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\right\}$ ve $L = \{e_1, e_2, e_3\}$ için $M(f, T, L)$ matrisini ve $r(M(2f, T, L))$ değerini bulunuz.

Çözüm: $f\left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right) = (2,0,0) = 2e_1 + 0e_2 + 0e_3$

$f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = (1,0,1) = 1e_1 + 0e_2 + 1e_3$

$f\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = (1,1,0) = 1e_1 + 1e_2 + 0e_3$

$f\left(\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\right) = (1,-1,-1) = 1e_1 + (-1)e_2 + (-1)e_3$,

$$\Rightarrow M(f,T,L) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

$Çekf = \left\{\begin{bmatrix} x & y \\ z & t \end{bmatrix} \in M_2(\mathbb{R}) \mid x+y=0,\ y+z=0,\ z+t=0\right\}$

$= \left\{\begin{bmatrix} x & y \\ z & t \end{bmatrix} \in M_2(\mathbb{R}) \mid x=z,\ y=-z,\ t=-z\right\}$

$= \left\{\begin{bmatrix} z & -z \\ z & -z \end{bmatrix} \mid z \in \mathbb{R}\right\} = \left\{z\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \mid z \in \mathbb{R}\right\} \Rightarrow Boy\,Çekf = 1$

$\Rightarrow Boy\,Resf = 4-1 = 3 \Rightarrow r(f) = 3$

$\Rightarrow r(M(f,T,L)) = 3$.

Teo: $r(f) = r(M(f,T,S))$

Teo: $\alpha \neq 0 \Rightarrow r(A) = r(\alpha A)$

Teo: $M(\alpha f, T, S) = \alpha M(f,T,S)$

$\Rightarrow r(M(2f,T,L)) = r(2M(f,T,L)) = r(M(f,T,L)) = 3$

Not: $M(2f,T,L)$ eşolon form matrisine dönüştürülerek de rank belirlenebilir.

(15p) S.2) $a, b \in \mathbb{R}$ ve $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & a \\ b & 0 & 6 \end{bmatrix}$ olmak üzere, $r(A)$ değerini ve A matrisinin regüler olma durumları için A^{-1} matrisini bulunuz.

Teo: $A \sim B \Rightarrow r(A) = r(B)$

Teo: A satır eş. form. mat. sıfırdan farklı satırların sayısı k ise $r(A) = k$.

Çözüm: $A \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & a-6 \\ 0 & 0 & 6-3b \end{bmatrix} = B$

i) $6 - 3b = 0 \Rightarrow B$ satır eşolan form mat.
$\Rightarrow r(B) = 2.$
$\Rightarrow r(A) = 2.$

ii) $b \neq 2 \Rightarrow A \sim B \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & a-6 \\ 0 & 0 & 1 \end{bmatrix} = C \Rightarrow r(A) = r(B) = r(C) = 3$

Sonuç: i) $b = 2$, $a \in \mathbb{R}$ için $r(A) = 2$.
ii) $b \neq 2$, " " $r(A) = 3 \Rightarrow A$ regüler.

$$[A : I] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & a-6 & -2 & 1 & 0 \\ 0 & 0 & 6-3b & -b & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & a-6 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{b}{3b-6} & 0 & \frac{1}{6-3b} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-2}{b-2} & 0 & \frac{1}{b-2} \\ 0 & 1 & 0 & \frac{12-ab}{3b-6} & 1 & \frac{a-6}{3b-6} \\ 0 & 0 & 1 & \frac{b}{3b-6} & 0 & \frac{1}{6-3b} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} \frac{2}{2-b} & 0 & \frac{1}{b-2} \\ \frac{12-ab}{3b-6} & 1 & \frac{a-6}{3b-6} \\ \frac{b}{3b-6} & 0 & \frac{1}{6-3b} \end{bmatrix}$$

Not: A^{-1} farklı yöntemlerlede belirlenebilir.

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(15p) S.3) $B = \begin{bmatrix} -1 & 2 & -7 & 6 \\ 1 & -2 & 4 & -3 \\ 1 & -2 & 6 & -5 \end{bmatrix}$ ve $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ için $PBQ = C$ olacak şekilde P ve Q tersinir matrislerini bulunuz.

Çözüm:

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -1 & 2 & -7 & 6 \\ 0 & 1 & 0 & 1 & -2 & 4 & -3 \\ 0 & 0 & 1 & 1 & -2 & 6 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -1 & 2 & -7 & 6 \\ 1 & 1 & 0 & 0 & 0 & -3 & 3 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|cccc} -1 & 0 & 0 & 1 & -2 & 7 & -6 \\ -1 & 0 & -1 & 0 & 0 & 1 & -1 \\ -2 & 1 & -3 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|cccc} 6 & 0 & 7 & 1 & -2 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 & -1 \\ -2 & 1 & -3 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$P = \begin{bmatrix} 6 & 0 & 7 \\ -1 & 0 & -1 \\ -2 & 1 & -3 \end{bmatrix}, \quad P.B = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \approx \left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \approx \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$Q = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ için } \quad P.B.Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ dır.}$$

Not: Satır ve Sütun işlemlerindeki tercihlere göre P, Q regüler matrislerinde farklılıklar olabilir.

S.4) (10p) a) $D \in M_n(\mathbb{R})$ diyagonalleştirilebilir matris ise D^T diyagonalleştirilebilir matris midir?

Çözüm: $\exists P$ regüler $P.A.P^{-1} = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \Rightarrow (P^{-1})^T . A^T . P^T = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$

$\Rightarrow (P^T)^{-1} . A^T . [(P^T)^{-1}]^{-1} = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \Rightarrow A^T$ diyagonalleştirilebilir m.

(10p) b) $E \in M_n(\mathbb{R})$ ortagonal matris ise $\det E$ nin alabileceği değerleri bulunuz.

Çözüm $E^T.E = I \Rightarrow \det E^T.E = \det I$

$\Rightarrow \det E^T . \det E = 1$

$\Rightarrow \det E . \det E = 1$

$\Rightarrow \det E \in \{1, -1\}$

Teo $\det AB = \det A . \det B$

Teo $\det A^T = \det A$

S.5) $X = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ ve $Y = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ matrisleri verilsin.

(10p) a) X ile Y benzer matrisler midir? $P = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ regüler nol.

Çözüm: $\begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} . \begin{bmatrix} x & y \\ z & t \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix}$

$\Rightarrow x + 2y = 0 \quad \Rightarrow x = -2y$

$z + 2t = -z \qquad z + t = 0, \ P = \begin{bmatrix} -2y & y \\ z & -z \end{bmatrix}, \quad z = y = 1, \ x = -2, \ t = -1$

$-z - 2t = -t$ Regüler matris ve

$P . X P^{-1} = Y$ dir. $\Rightarrow X$ ile Y benzer matrislerdir.

(10p) b) X ile Y eş matrisler midir?

Çözüm: X simetrik matris değil
Y " " $\Big\} \Rightarrow X$ ile Y eş matrisler değil.

Teo: A ile B eş m., A sim. m ise B simetrik m.

Not: Tanım kullanılarak da çözülebilir.

(15p) S.6) $2x_1 + 5x_2 - ax_3 = 1$
$x_1 + 3x_2 + 3x_3 = 0$
$4x_1 + 11x_2 + 5x_3 = 1$

$$A = \begin{bmatrix} 2 & 5 & -a \\ 1 & 3 & 3 \\ 4 & 11 & 5 \end{bmatrix}, \quad [A:b] = \left[\begin{array}{ccc|c} 2 & 5 & -a & 1 \\ 1 & 3 & 3 & 0 \\ 4 & 11 & 5 & 1 \end{array}\right]$$

lineer denklem sisteminin çözüm kümelerini $a \in \mathbb{R}$ değerlerine göre ekli matris sistemi üzerinde satır işlemleri yaparak belirleyiniz.

Çözüm:
$$\left[\begin{array}{ccc|c} 2 & 5 & -a & 1 \\ 1 & 3 & 3 & 0 \\ 4 & 11 & 5 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & -1 & -a-6 & 1 \\ 0 & -1 & -7 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & 1 & 7 & -1 \\ 0 & 1 & a+6 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & a-1 & 0 \end{array}\right] = C$$

i) $a = 1$ ise C satır eşolon form matr. $\Rightarrow r(A) = r([A:b]) = 2$
sonsuz çözüm vardır.

$x_2 + 7x_3 = -1 \Rightarrow x_2 = -1 - 7x_3$

$x_1 + 3x_2 + 3x_3 = 0 \quad x_1 = -3x_3 - 3(-1-7x_3) \Rightarrow x_1 = 3 + 18x_3$

$$\left\{ \begin{bmatrix} 3+18x_3 \\ -1-7x_3 \\ x_3 \end{bmatrix} \middle| \, x_3 \in \mathbb{R} \right\}$$ çözüm kümesi

ii) $a \neq 1 \Rightarrow C \sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$x_3 = 0$
$x_2 + 7x_3 = -1 \Rightarrow x_2 = -1$
$x_1 + 3x_2 + 3x_3 = 0$
$\Rightarrow x_1 = 3$

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$$ çözüm kümesi.