Mechanical Vibrations

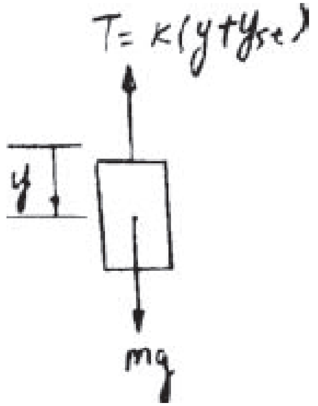
Free vibration of SDOF systems: Review problems

Problem 1

22–1. A spring is stretched 175 mm by an 8-kg block. If the block is displaced 100 mm downward from its equilibrium position and given a downward velocity of 1.50 m/s, determine the differential equation which describes the motion. Assume that positive displacement is downward. Also, determine the position of the block when $t = 0.22$ s.

$+\downarrow \Sigma F_y = ma_y;\qquad mg - k(y + y_{st}) = m\ddot{y} \qquad \text{where } ky_{st} = mg$

$$\ddot{y} + \frac{k}{m}y = 0$$



Hence $\qquad p = \sqrt{\dfrac{k}{m}} \qquad$ Where $k = \dfrac{8(9.81)}{0.175} = 448.46 \text{ N/m}$

$$= \sqrt{\frac{448.46}{8}} = 7.487$$

$\therefore \qquad \ddot{y} + (7.487)^2 y = 0 \qquad \ddot{y} + 56.1y = 0$ **Ans.**

The solution of the above differential equation is of the form:

$y = A \sin pt + B \cos pt$ **(1)**

$v = \dot{y} = Ap \cos pt - Bp \sin pt$ **(2)**

At $t = 0$, $y = 0.1$ m and $v = v_0 = 1.50$ m/s

From Eq. (1) $\qquad 0.1 = A \sin 0 + B \cos 0 \qquad B = 0.1$ m

Problem 2

22–2. A spring has a stiffness of 800 N/m. If a 2-kg block is attached to the spring, pushed 50 mm above its equilibrium position, and released from rest, determine the equation that describes the block's motion. Assume that positive displacement is downward.

$$p = \sqrt{\frac{k}{m}} = \sqrt{\frac{800}{2}} = 20$$

$$x = A \sin pt + B \cos pt$$

$x = -0.05$ m when $t = 0$,

$$-0.05 = 0 + B; \qquad B = -0.05$$

$$v = Ap \cos pt - Bp \sin pt$$

$v = 0$ when $t = 0$,

$$0 = A(20) - 0; \qquad A = 0$$

Thus,

$$x = -0.05 \cos (20t)$$ **Ans.**

Problem 3

*22–8. A 2-kg block is suspended from a spring having a stiffness of 800 N/m. If the block is given an upward velocity of 2 m/s when it is displaced downward a distance of 150 mm from its equilibrium position, determine the equation which describes the motion. What is the amplitude of the motion? Assume that positive displacement is downward.

$$p = \sqrt{\frac{k}{m}} = \sqrt{\frac{800}{2}} = 20$$

$$x = A \sin pt + B \cos pt$$

$$x = 0.150 \text{ m when } t = 0,$$

$$0.150 = 0 + B; \qquad B = 0.150$$

$$v = Ap \cos pt - Bp \sin pt$$

$$v = -2 \text{ m/s when } t = 0,$$

$$-2 = A(20) - 0; \qquad A = -0.1$$

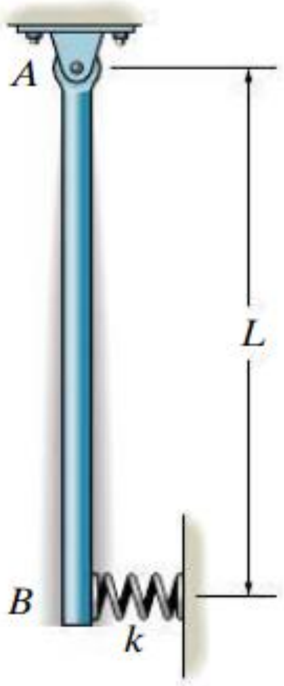
Thus,

$$x = 0.1 \sin (20t) + 0.150 \cos (20t) \qquad \textbf{Ans.}$$

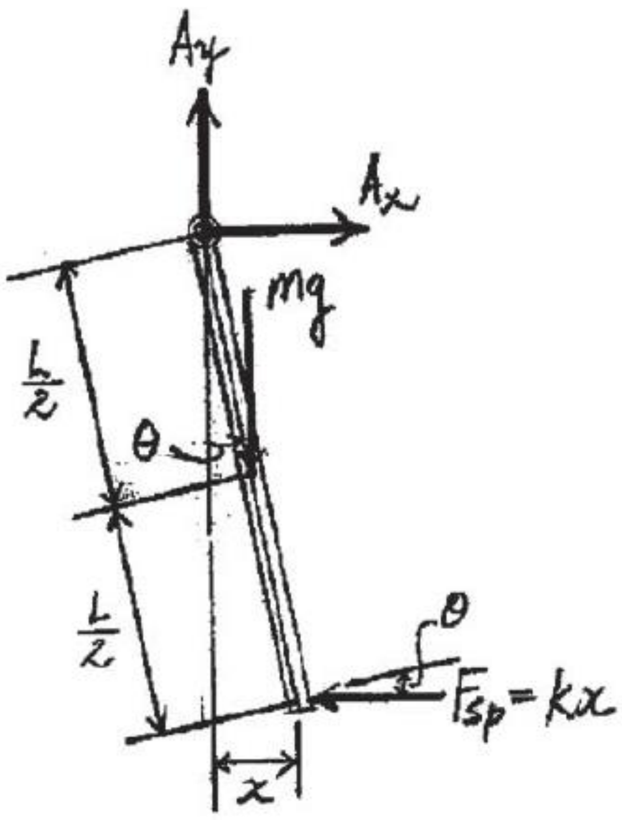
$$C = \sqrt{A^2 + B^2} = \sqrt{(0.1)^2 + (0.150)^2} = 0.180 \text{ m} \qquad \textbf{Ans.}$$

Problem 4

22–10. The uniform rod of mass m is supported by a pin at A and a spring at B. If B is given a small sideward displacement and released, determine the natural period of vibration.



Prob. 22–10



(a)

SOLUTION

Equation of Motion. The mass moment of inertia of the rod about A is $I_A = \frac{1}{3}mL^2$. Referring to the FBD. of the rod, Fig. *a*,

$$\circlearrowleft + \Sigma M_A = I_A\alpha; \qquad -mg\left(\frac{L}{2}\sin\theta\right) - (kx\cos\theta)(L) = \left(\frac{1}{3}mL^2\right)\alpha$$

However; $\quad x = L\sin\theta$. Then

$$\frac{-mgL}{2}\sin\theta - kL^2\sin\theta\cos\theta = \frac{1}{3}mL^2\alpha$$

Using the trigonometry identity $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\frac{-mgL}{2}\sin\theta - \frac{KL^2}{2}\sin 2\theta = \frac{1}{3}mL^2\alpha$$

Here since θ is small $\sin\theta \simeq \theta$ and $\sin 2\theta \simeq 2\theta$. Also $\alpha = \ddot{\theta}$. Then the above equation becomes

$$\frac{1}{3}mL^2\ddot{\theta} + \left(\frac{mgL}{2} + kL^2\right)\theta = 0$$

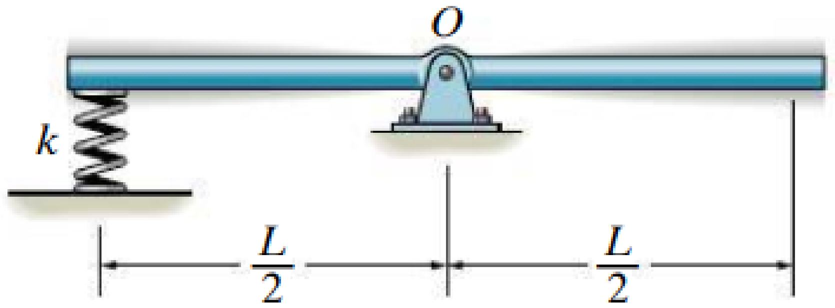
$$\ddot{\theta} + \frac{3mg + 6kL}{2mL}\theta = 0$$

Comparing to that of the Standard form, $\omega_n = \sqrt{\frac{3mg + 6kL}{2mL}}$. Then

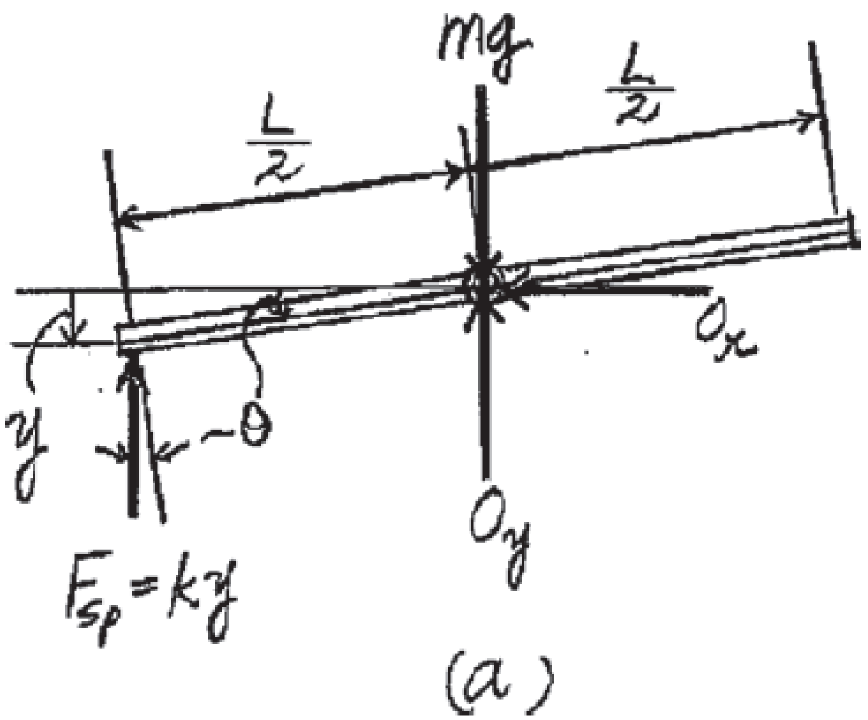
$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{2mL}{3mg + 6kL}} \qquad \textbf{Ans.}$$

Problem 5

22–13. Determine the natural period of vibration of the uniform bar of mass m when it is displaced downward slightly and released.



Prob. 22–13



SOLUTION

Equation of Motion. The mass moment of inertia of the bar about O is $I_0 = \frac{1}{12}mL^2$. Referring to the FBD of the rod, Fig. *a*,

$$\circlearrowleft + \Sigma M_0 = I_0\alpha; \quad -ky\cos\theta\left(\frac{L}{2}\right) = \left(\frac{1}{12}mL^2\right)\alpha$$

However, $y = \frac{L}{2}\sin\theta$. Then

$$-k\left(\frac{L}{2}\sin\theta\right)\cos\theta\left(\frac{L}{2}\right) = \frac{1}{12}mL^2\alpha$$

Using the trigonometry identity $\sin 2\theta = 2\sin\theta\cos\theta$, we obtain

$$\frac{1}{12}mL^2\alpha + \frac{kL^2}{8}\sin 2\theta = 0$$

Here since θ is small, $\sin 2\theta \simeq 2\theta$. Also, $\alpha = \ddot{\theta}$. Then the above equation becomes

$$\frac{1}{12}mL^2\ddot{\theta} + \frac{kL^2}{4}\theta = 0$$

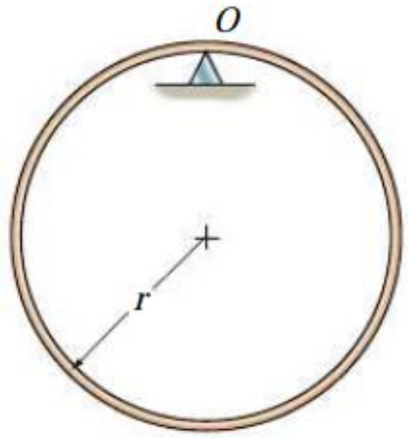
$$\ddot{\theta} + \frac{3k}{m}\theta = 0$$

Comparing to that of the Standard form, $\omega_n = \sqrt{\frac{3k}{m}}$. Then

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{3k}}$$ **Ans.**

Problem 6

22–15. The thin hoop of mass m is supported by a knife-edge. Determine the natural period of vibration for small amplitudes of swing.



Prob. 22–15

SOLUTION

$$I_O = mr^2 + mr^2 = 2mr^2$$

$$\circlearrowleft + \sum M_O = I_O\alpha; \qquad -mgr\theta = (2mr^2)\ddot{\theta}$$

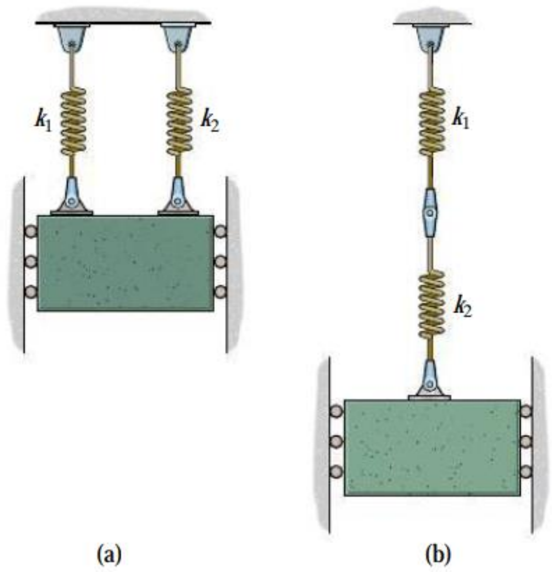
$$\ddot{\theta} + \left(\frac{g}{2r}\right)\theta = 0$$

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{2r}{g}}$$ **Ans.**

Problem 7

***22–16.** A block of mass m is suspended from two springs having a stiffness of k_1 and k_2, arranged a) parallel to each other, and b) as a series. Determine the equivalent stiffness of a single spring with the same oscillation characteristics and the period of oscillation for each case.

22–17. The 15-kg block is suspended from two springs having different stiffnesses and arranged a) parallel to each other, and b) as a series. If the natural periods of oscillation of the parallel system and series system are observed to be 0.5 s and 1.5 s, respectively, determine the spring stiffnesses k_1 and k_2.



Probs. 22–16/17

SOLUTION

(a) When the springs are arranged in parallel, the equivalent spring stiffness is

$$k_{eq} = k_1 + k_2 \qquad \textbf{Ans.}$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 + k_2}{m}}$$

Thus, the period of oscillation of the system is

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\dfrac{k_1 + k_2}{m}}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}} \qquad \textbf{Ans.}$$

(b) When the springs are arranged in a series, the equivalent stiffness of the system can be determined by equating the stretch of both spring systems subjected to the same load F.

$$\frac{F}{k_1} + \frac{F}{k_2} = \frac{F}{k_{eq}}$$

$$\frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{k_{eq}}$$

$$\frac{k_2 + k_1}{k_1 k_2} = \frac{1}{k_{eq}}$$

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$ **Ans.**

The natural frequency of the system is

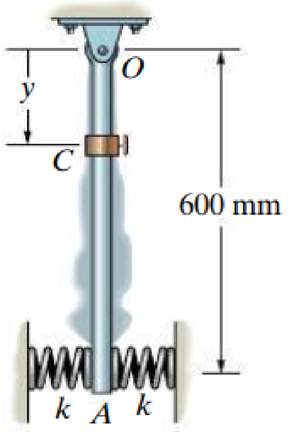
$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{\left(\frac{k_1 k_2}{k_2 + k_1}\right)}{m}}$$

Thus, the period of oscillation of the system is

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{\left(\frac{k_1 k_2}{k_2 + k_1}\right)}{m}}} = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$ **Ans.**

Problem 8

22–19. The slender rod has a mass of 0.2 kg and is supported at O by a pin and at its end A by two springs, each having a stiffness $k = 4\ \mathrm{N/m}$. The period of vibration of the rod can be set by fixing the 0.5-kg collar C to the rod at an appropriate location along its length. If the springs are originally unstretched when the rod is vertical, determine the position y of the collar so that the natural period of vibration becomes $\tau = 1$ s. Neglect the size of the collar.



Prob. 22–19

SOLUTION

Moment of inertia about O:

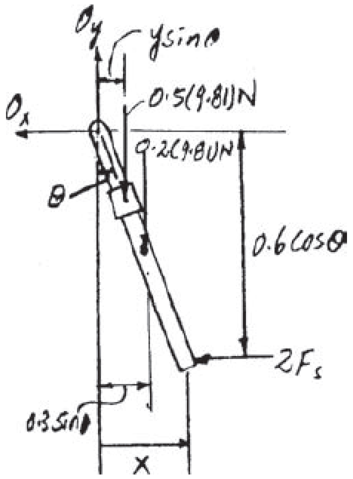
$$I_O = \frac{1}{3}(0.2)(0.6)^2 + 0.5y^2 = 0.024 + 0.5y^2$$

Each spring force $F_s = kx = 4x$.

$$\circlearrowleft +\Sigma M_O = I_O\,\alpha; \qquad -2(4x)(0.6\cos\theta) - 0.2(9.81)(0.3\sin\theta)$$

$$-0.5(9.81)(y\sin\theta) = (0.024 + 0.5y^2)\,\ddot{\theta}$$

$$-4.8x\cos\theta - (0.5886 + 4.905y)\sin\theta = (0.024 + 0.5y^2)\ddot{\theta}$$



However, for small displacement $x = 0.6\theta$, $\sin\theta \approx \theta$ and $\cos\theta = 1$. Hence

$$\ddot{\theta} + \frac{3.4686 + 4.905y}{0.024 + 0.5y^2}\theta = 0$$

From the above differential equation, $p = \sqrt{\dfrac{3.4686 + 4.905y}{0.024 + 0.5y^2}}$.

$$\tau = \frac{2\pi}{p}$$

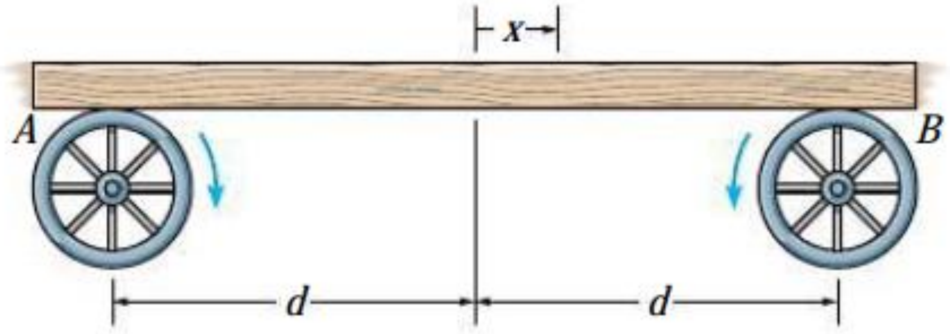
$$1 = \frac{2\pi}{\sqrt{\dfrac{3.4686 + 4.905y}{0.024 + 0.5y^2}}}$$

$$19.74y^2 - 4.905y - 2.5211 = 0$$

$$y = 0.503\text{ m} = 503\text{ mm}$$ **Ans.**

Problem 9

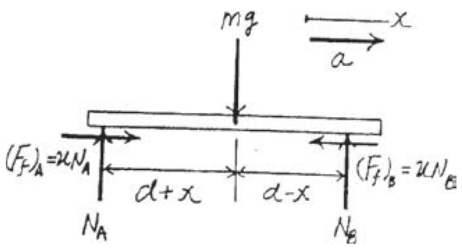
***22–20.** A uniform board is supported on two wheels which rotate in opposite directions at a constant angular speed. If the coefficient of kinetic friction between the wheels and board is μ, determine the frequency of vibration of the board if it is displaced slightly, a distance x from the midpoint between the wheels, and released.



Prob. 22–20

SOLUTION

Freebody Diagram: When the board is being displaced x to the right, the *restoring force* is due to the unbalance friction force at A and B $\left[(F_f)_B > (F_f)_A\right]$.



Equation of Motion:

$$\circlearrowleft + \Sigma M_A = \Sigma (M_A)_k; \qquad N_B\,(2d) - mg(d + x) = 0$$

$$N_B = \frac{mg(d + x)}{2d}$$

$$+\uparrow \Sigma F_y = m(a_G)_y; \qquad N_A + \frac{mg(d + x)}{2d} - mg = 0$$

$$N_A = \frac{mg(d - x)}{2d}$$

$$\xrightarrow{+} \Sigma F_x = m(a_G)_x; \qquad \mu\left[\frac{mg(d - x)}{2d}\right] - \mu\left[\frac{mg(d + x)}{2d}\right] = ma$$

$$a + \frac{\mu g}{d}x = 0 \qquad \textbf{(1)}$$

Kinematics: Since $a = \dfrac{d^2x}{dt^2} = \ddot{x}$, then substitute this value into Eq.(1), we have

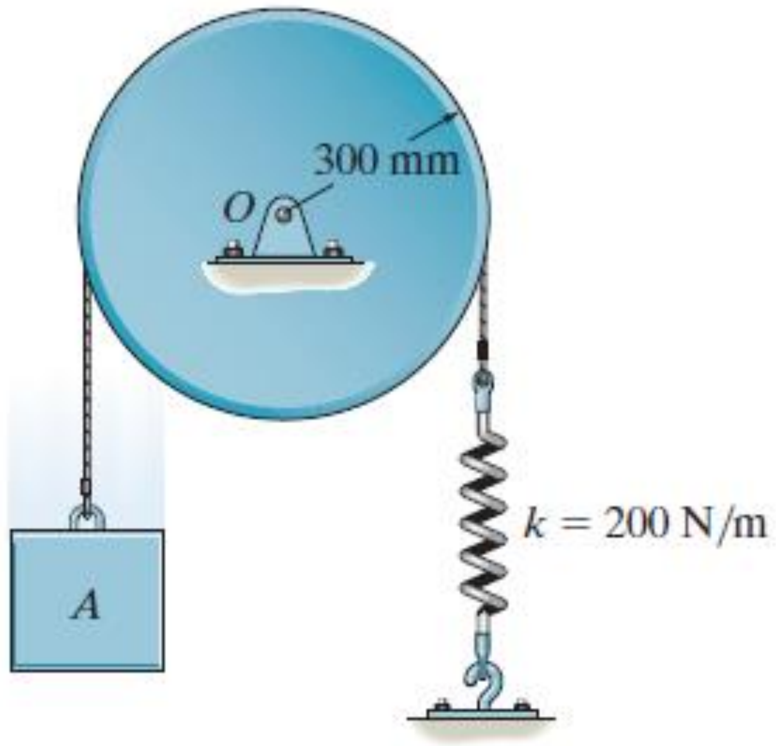
$$\ddot{x} + \frac{\mu g}{d}x = 0 \qquad \textbf{(2)}$$

From Eq.(2), $\omega_n{}^2 = \dfrac{\mu g}{d}$, thus, $\omega_n = \sqrt{\dfrac{\mu g}{d}}$. Applying Eq. 22–4, we have

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\mu g}{d}} \qquad \textbf{Ans.}$$

Problem 10

22–23. The 20-kg disk, is pinned at its mass center O and supports the 4-kg block A. If the belt which passes over the disk is not allowed to slip at its contacting surface, determine the natural period of vibration of the system.



SOLUTION

Equation of Motion. The mass moment of inertia of the disk about its mass center O is $I_0 = \frac{1}{2}mr^2 = \frac{1}{2}(20)(0.3^2) = 0.9\ \text{kg}\cdot\text{m}^2$. When the disk undergoes a small angular displacement θ, the spring stretches further by $s = r\theta = 0.3\theta$. Thus, the total stretch is $y = y_{st} + 0.3\theta$. Then $F_{sp} = ky = 200(y_{st} + 0.3\theta)$. Referring to the FBD and kinetic diagram of the system, Fig. a,

$$\circlearrowleft +\Sigma M_0 = \Sigma(\mu_k)_0; \quad 4(9.81)(0.3) - 200(y_{st} + 0.3\theta)(0.3) = 0.90\alpha + 4[\alpha(0.3)](0.3)$$

$$11.772 - 60y_{st} - 18\theta = 1.26\alpha \qquad \textbf{(1)}$$

When the system is in equilibrium, $\theta = 0°$. Then

$$\circlearrowleft +\Sigma M_0 = 0; \qquad 4(9.81)(0.3) - 200(y_{st})(0.3) = 0$$

$$60y_{st} = 11.772$$

Substitute this result into Eq. (1), we obtain

$$-18\theta = 1.26\alpha$$

$$\alpha + 14.28571\theta = 0$$

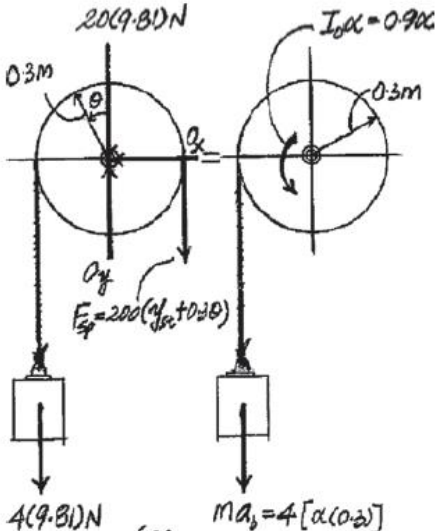
Since $\alpha = \ddot{\theta}$, the above equation becomes

$$\ddot{\theta} + 14.28571\theta = 0$$

Comparing to that of standard form, $\omega_n = \sqrt{14.2857} = 3.7796$ rad/s.

Thus,

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.7796} = 1.6623\ \text{s} = 1.66\ \text{s} \qquad \textbf{Ans.}$$

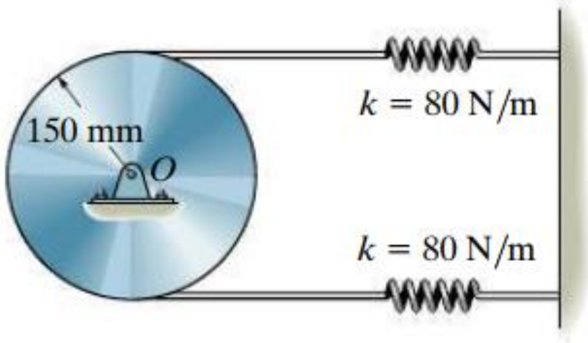


(a)

Problem 11

*22–24.** The 10-kg disk is pin connected at its mass center. Determine the natural period of vibration of the disk if the springs have sufficient tension in them to prevent the cord from slipping on the disk as it oscillates. *Hint:* Assume that the initial stretch in each spring is δ_O.

22–25. If the disk in Prob. 22–24 has a mass of 10 kg, determine the natural frequency of vibration. *Hint:* Assume that the initial stretch in each spring is δ_O.



Probs. 22–24/25

SOLUTION

Equation of Motion. The mass moment of inertia of the disk about its mass center O is $I_0 = \frac{1}{2}Mr^2 = \frac{1}{2}(10)(0.15^2) = 0.1125\ \text{kg}\cdot\text{m}^2$. When the disk undergoes a small angular displacement θ, the top spring stretches further but the stretch of the spring is being reduced both by $s = r\theta = 0.15\theta$. Thus, $(F_{sp})_t = Kx_t = 80(\delta_0 - 0.15\theta)$ and $(F_{sp})_b = 80(\delta_0 - 0.15\theta)$. Referring to the FBD of the disk, Fig. *a*,

$$\circlearrowleft +\Sigma M_0 = I_0\alpha; \qquad -80(\delta_0 + 0.15\theta)(0.15) + 80(\delta_0 - 0.15\theta)(0.15) = 0.1125\alpha$$

$$-3.60\theta = 0.1125\alpha$$

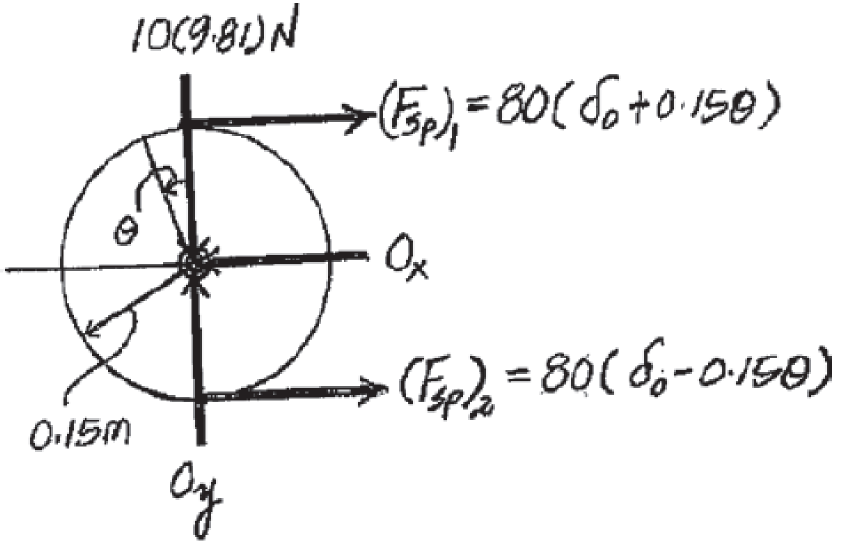
$$\alpha + 32\theta = 0$$

Since $\alpha = \ddot{\theta}$, this equation becomes

$$\ddot{\theta} + 32\theta = 0$$

Comparing to that of standard form, $\omega_n = \sqrt{32}$ rad/s. Then

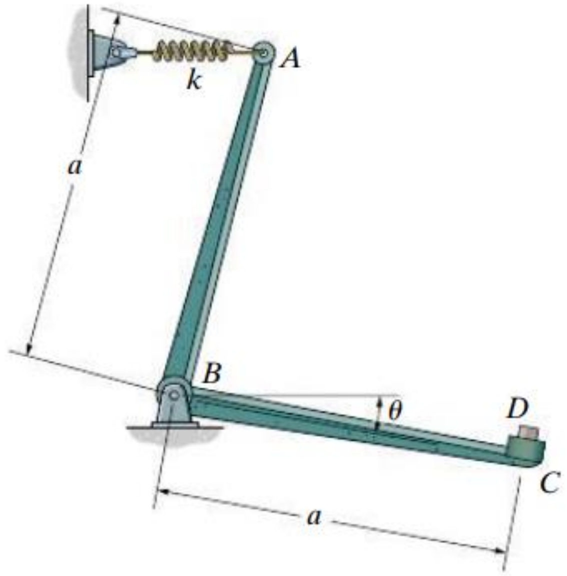
$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{32}} = 1.1107\ \text{s} = 1.11\ \text{s}$$ **Ans.**



(a)

Problem 12

22–27. If a block D of negligible size and of mass m is attached at C, and the bell crank of mass M is given a small angular displacement of θ, the natural period of oscillation is τ_1. When D is removed, the natural period of oscillation is τ_2. Determine the bell crank's radius of gyration about its center of mass, pin B, and the spring's stiffness k. The spring is unstrectched at $\theta = 0°$, and the motion occurs in the *horizontal plane*.



Prob. 22–27

SOLUTION

Equation of Motion: When the bell crank rotates through a small angle θ, the spring stretches $s = a\theta$. Thus, the force in the spring is $F_{sp} = ks = k(a\theta)$. The mass moment of inertia of the bell crank about its mass center B is $I_B = Mk_B^2$. Referring to the free-body diagram of the bell crank shown in Fig. *a*,

$$\circlearrowleft +\Sigma M_B = I_B\alpha; \qquad -k(a\theta)\cos\theta(a) = Mk_B^2\ddot{\theta}_B$$

$$\ddot{\theta} + \frac{ka^2}{Mk_B^2}(\cos\theta)\theta = 0 \qquad \textbf{(1)}$$

Since θ is very small, $\cos\theta \cong 1$. Then Eq.(1) becomes

$$\ddot{\theta} + \frac{ka^2}{Mk_B^2}\theta = 0$$

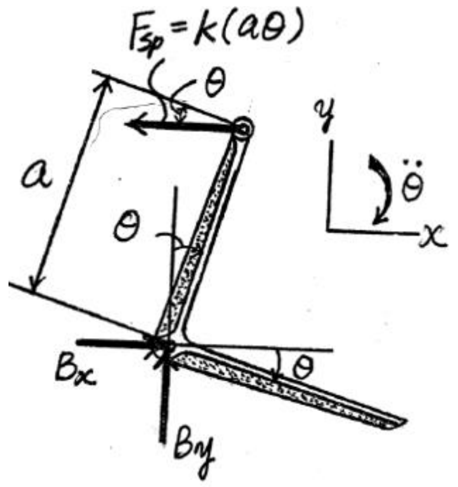
Since the bell crank rotates about point B, $a_C = \alpha r_{BC} = \ddot{\theta}(a)$. Referring to the free-body diagram shown in Fig. *b*,

$$\circlearrowleft +\Sigma M_B = \Sigma(M_k)_B; \qquad -k(a\theta)\cos\theta(a) = Mk_B^2\ddot{\theta} + m\left[\ddot{\theta}(a)\right](a)$$

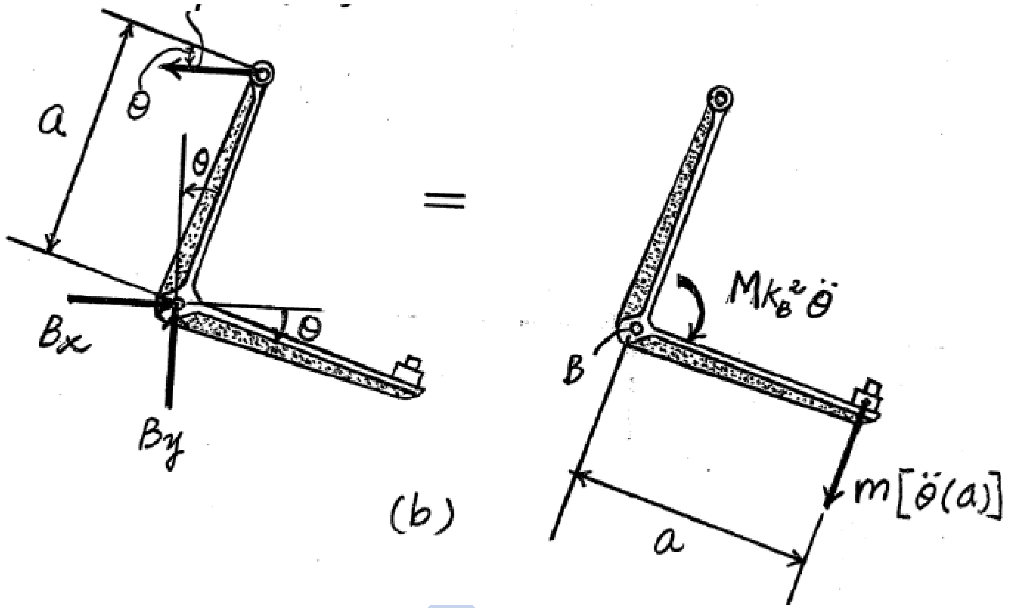
$$\ddot{\theta} + \frac{ka^2}{Mk_B^2 + ma^2}(\cos\theta)\theta = 0 \qquad \textbf{(2)}$$

Again, $\cos\theta \cong 1$, since θ is very small. Thus, Eq. (2) becomes

$$\ddot{\theta} + \frac{ka^2}{Mk_B^2 + ma^2}\theta = 0$$



(a)



(b)

Thus, the natural frequencies of the two oscillations are

$$(\omega_n)_2 = \sqrt{\frac{ka^2}{Mk_B^2}}$$

$$(\omega_n)_1 = \sqrt{\frac{ka^2}{Mk_B^2 + ma^2}}$$

The natural periods of the two oscillations are

$$\tau_2 = \frac{2\pi}{(\omega_n)_2} = 2\pi\sqrt{\frac{Mk_B^2}{ka^2}}$$

$$\tau_1 = \frac{2\pi}{(\omega_n)_1} = 2\pi\sqrt{\frac{Mk_B^2 + ma^2}{ka^2}}$$

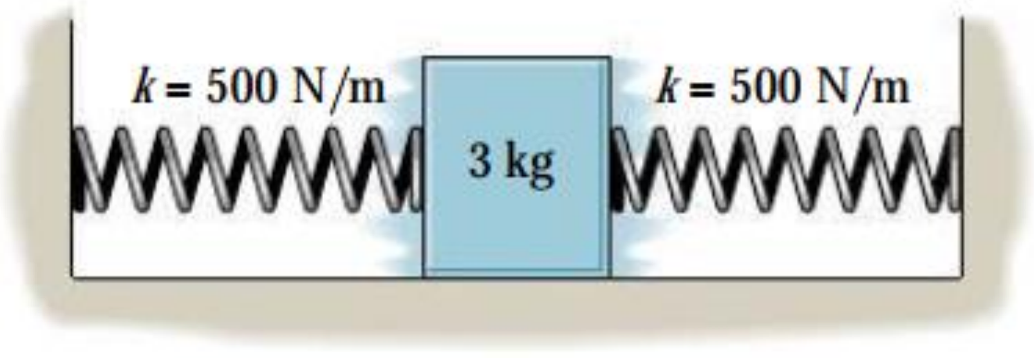
Solving,

$$k_B = a\sqrt{\frac{m}{M}\left(\frac{\tau_2^2}{\tau_1^2 - \tau_2^2}\right)} \qquad \textbf{Ans.}$$

$$k = \frac{4\pi^2}{\tau_1^2 - \tau_2^2}\, m \qquad \textbf{Ans.}$$

Problem 13

22–30. Determine the differential equation of motion of the 3-kg block when it is displaced slightly and released. The surface is smooth and the springs are originally unstretched.



Prob. 22–30

SOLUTION

$T + V = \text{const.}$

$$T = \frac{1}{2}(3)\dot{x}^2$$

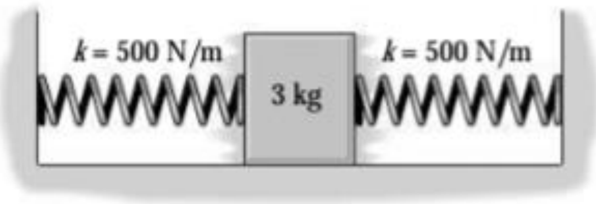
$$V = \frac{1}{2}(500)x^2 + \frac{1}{2}(500)x^2$$

$$T + V = 1.5\dot{x}^2 + 500x^2$$

$$1.5(2\dot{x})\,\ddot{x} + 1000x\dot{x} = 0$$

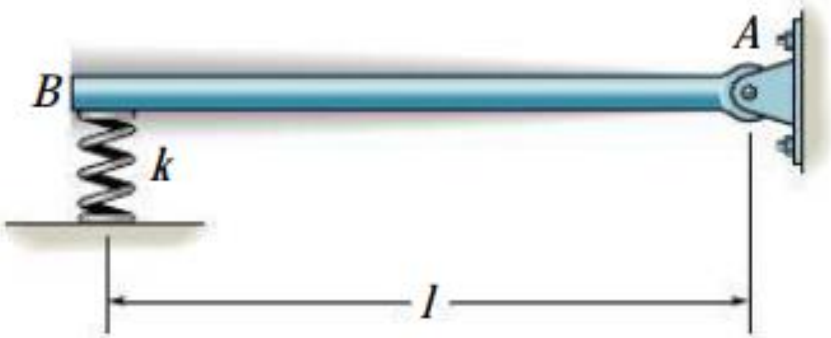
$$3\ddot{x} + 1000x = 0$$

$$\ddot{x} + 333x = 0 \qquad \textbf{Ans.}$$



Problem 14

22–31. The uniform rod of mass m is supported by a pin at A and a spring at B. If the end B is given a small downward displacement and released, determine the natural period of vibration.



Prob. 22–31

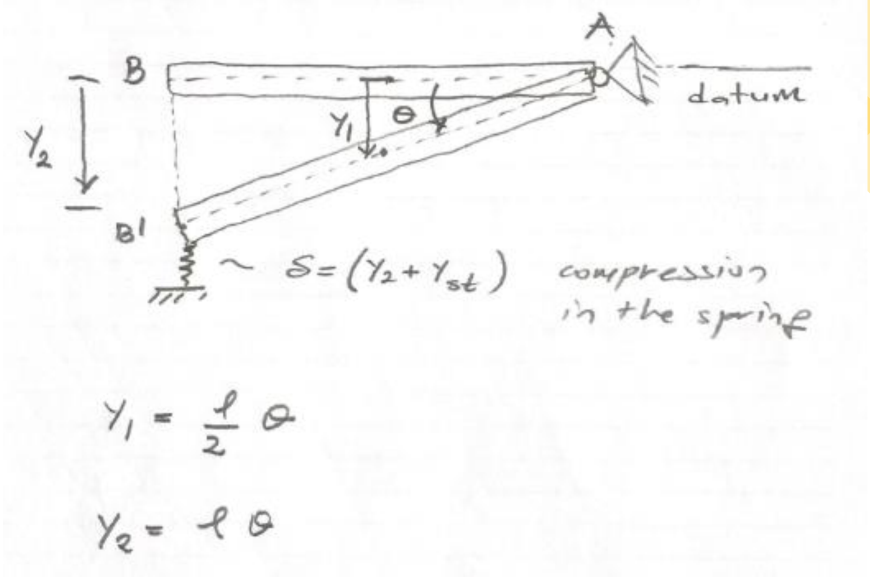
SOLUTION

$$T = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2$$

$$V = \frac{1}{2}k(y_{eq} + y_2)^2 - mgy_1$$

$$= \frac{1}{2}k(l\theta_{eq} + l\theta)^2 - mg\left(\frac{1}{2}\right)\theta$$

$$T + V = \frac{1}{6}ml^2\dot{\theta}^2 + \frac{1}{2}k(l\theta_{eq} + l\theta)^2 - mg\left(\frac{l\theta}{2}\right)$$



Time derivative

$$0 = \frac{1}{3}ml^2\dot{\theta}\ddot{\theta} + kl(\theta_{eq} + \theta)\dot{\theta} - mgl\frac{\dot{\theta}}{2}$$

For equilibrium

$$k(l\theta_{eq}) = mgl/2, \theta_{eq} = \frac{mg}{2k}$$

Thus,

$$0 = \frac{1}{3}ml\ddot{\theta} + k\theta$$

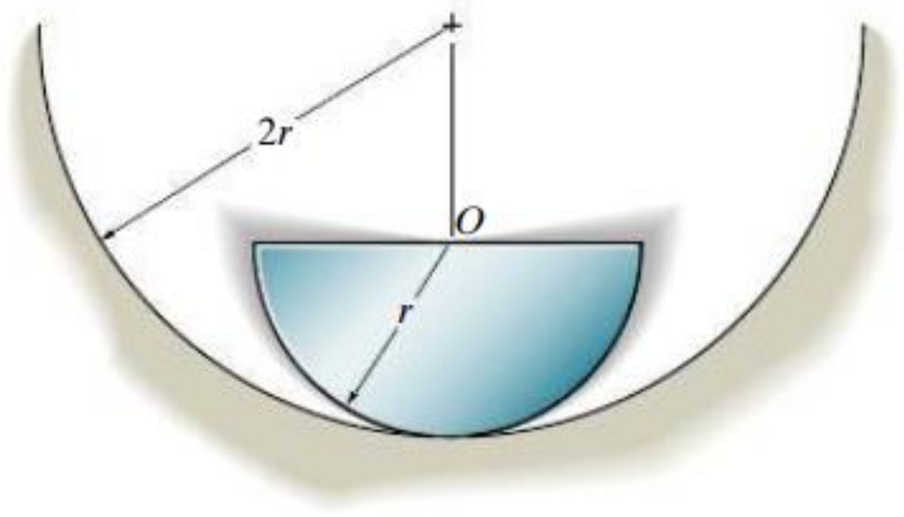
$$\ddot{\theta} + (3k/m)\theta = 0$$

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{3k}}$$

Ans.

Problem 15

*22–32. The semicircular disk has a mass m and radius r, and it rolls without slipping in the semicircular trough. Determine the natural period of vibration of the disk if it is displaced slightly and released. *Hint:* $I_O = \frac{1}{2}mr^2$.



Prob. 22–32

SOLUTION

$$AB = (2r - r)\cos\phi = r\cos\phi, \qquad BC = \frac{4r}{3\pi}\cos\theta$$

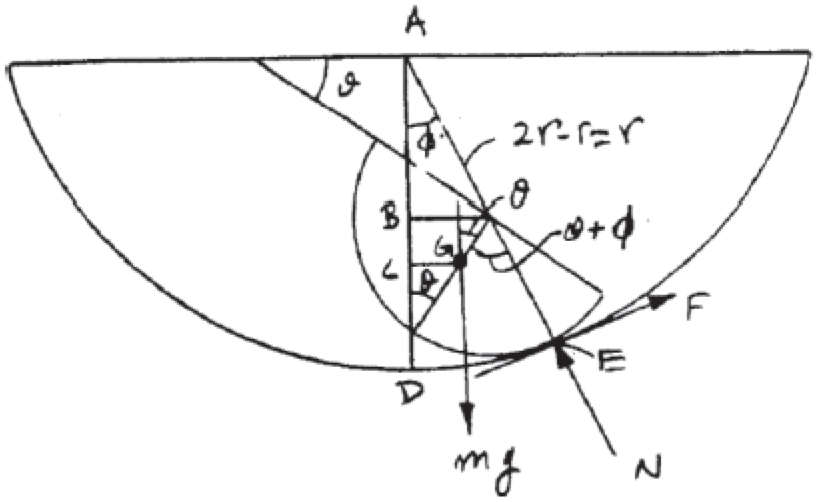
$$AC = r\cos\phi + \frac{4r}{3\pi}\cos\theta, \qquad DE = 2r\phi = r(\theta + \phi)$$

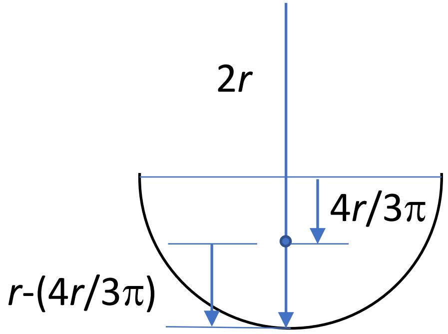
$$\phi = \theta$$

$$AC = r\left(1 + \frac{4}{3\pi}\right)\cos\theta$$

Thus, the change in elevation of G is

$$h = 2r - \left(r - \frac{4r}{3\pi}\right) - AC = r\left(1 + \frac{4}{3\pi}\right)(1 - \cos\theta)$$





Since no slipping occurs,

$$v_G = \dot{\theta}\left(r - \frac{4r}{3\pi}\right)$$

$$I_G = I_O - m\left(\frac{4r}{3\pi}\right)^2 = \left(\frac{1}{2} - \left(\frac{4}{3\pi}\right)^2\right)mr^2$$

$$T = \frac{1}{2}m\dot{\theta}^2r^2\left(1 - \frac{4}{3\pi}\right)^2 + \frac{1}{2}\left(\frac{1}{2} - \left(\frac{4}{3\pi}\right)^2\right)mr^2\dot{\theta}^2 = \frac{1}{2}mr^2\left(\frac{3}{2} - \frac{8}{3\pi}\right)\dot{\theta}^2$$

$$T + V = \frac{1}{2}mr^2\left(\frac{3}{2} - \frac{8}{3\pi}\right)\dot{\theta}^2 + mgr\left(1 + \frac{4}{3\pi}\right)(1 - \cos\theta)$$

$$0 = mr^2\left(\frac{3}{2} - \frac{8}{3\pi}\right)\dot{\theta}\,\ddot{\theta} + mgr\left(1 + \frac{4}{3\pi}\right)\sin\theta\dot{\theta}$$

$$\sin\theta \approx \theta$$

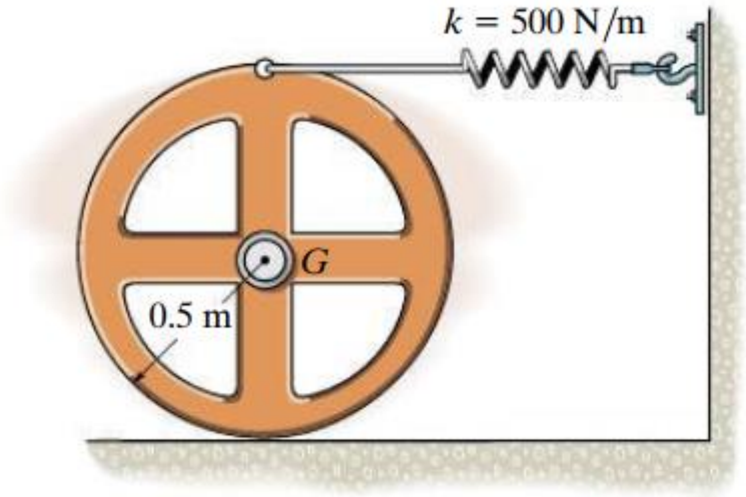
$$\ddot{\theta} + \frac{g\left(1 + \frac{4}{3\pi}\right)}{r\left(\frac{3}{2} - \frac{8}{3\pi}\right)}\theta = 0$$

$$\omega_n = 1.479\sqrt{\frac{g}{r}}$$

$$\tau = \frac{2\pi}{\omega_n} = 4.25\sqrt{\frac{r}{g}}$$ **Ans.**

Problem 16

22–33. If the 20-kg wheel is displaced a small amount and released, determine the natural period of vibration. The radius of gyration of the wheel is $k_G = 0.36$ m. The wheel rolls without slipping.



Prob. 22–33

SOLUTION

Energy Equation. The mass moment of inertia of the wheel about its mass center is $I_G = mk_G = 20(0.361)^2 = 2.592\ \text{kg}\cdot\text{m}^2$. Since the wheel rolls without slipping, $v_G = \omega r = \omega(0.5)$. Thus,

$$T = \frac{1}{2}I_G\omega^2 + \frac{1}{2}mv_G^2$$

$$= \frac{1}{2}(2.592)\omega^2 + \frac{1}{2}(20)[\omega(0.5)]^2$$

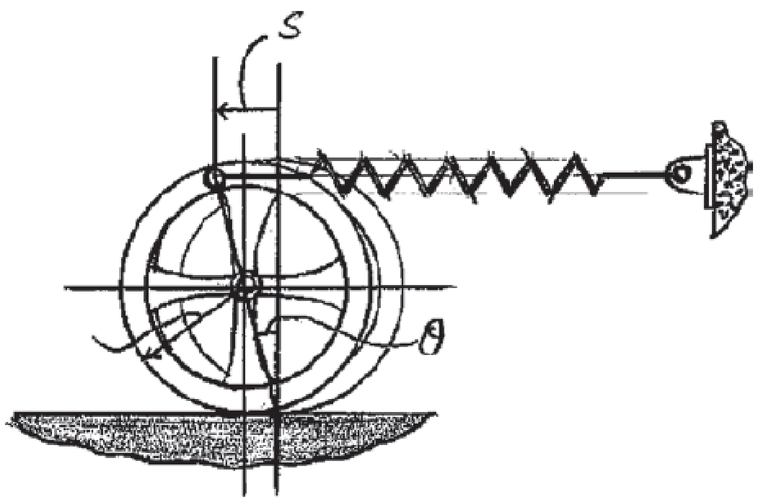
$$= 3.796\ \omega^2 = 3.796\dot{\theta}^2$$

When the disk undergoes a small angular displacement θ, the spring stretches $s = \theta(1) = \theta$, Fig. *a*. Thus, the elastic potential energy is

$$V_e = \frac{1}{2}ks^2 = \frac{1}{2}(500)\theta^2 = 250\theta^2$$

Thus, the total energy is

$$E = T + V = 3.796\dot{\theta}^2 + 250\theta^2$$



(a)

Time Derivative. Taking the time derivative of the above equation,

$$7.592\dot{\theta}\ddot{\theta} + 500\theta\dot{\theta} = 0$$

$$\dot{\theta}(7.592\ddot{\theta} + 500\theta) = 0$$

Since $\dot{\theta} \neq 0$, then

$$7.592\ddot{\theta} + 500\theta = 0$$

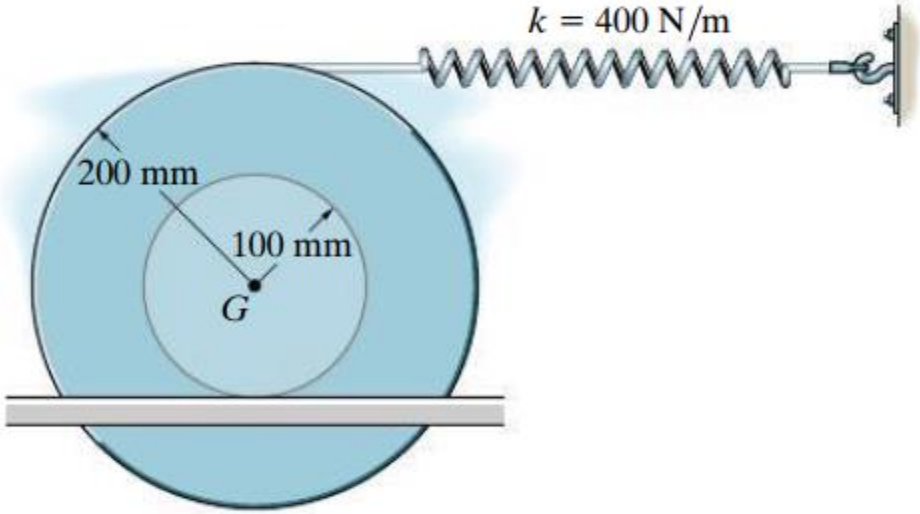
$$\ddot{\theta} + 65.8588\theta = 0$$

Comparing to that of standard form, $\omega_n = \sqrt{65.8588} = 8.1153$ rad/s. Thus,

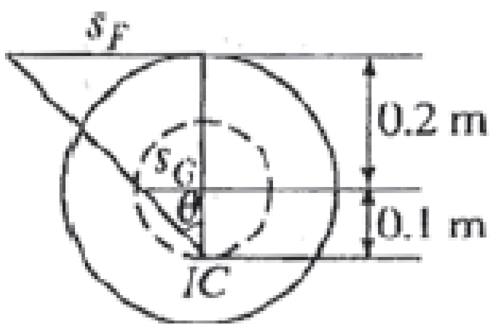
$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{8.1153} = 0.7742 \text{ s} = 0.774 \text{ s}$$ **Ans.**

Problem 17

22–34. Determine the differential equation of motion of the 3-kg spool. Assume that it does not slip at the surface of contact as it oscillates. The radius of gyration of the spool about its center of mass is $k_G = 125$ mm.



Prob. 22–34



SOLUTION

Kinematics: Since no slipping occurs, $s_G = 0.1\theta$ hence $s_F = \dfrac{0.3}{0.1}S_G = 0.3\theta$. Also,

$$v_G = 0.1\dot{\theta}.$$

$$E = T + V$$

$$E = \frac{1}{2}[(3)(0.125)^2]\dot{\theta}^2 + \frac{1}{2}(3)(0.1\dot{\theta})^2 + \frac{1}{2}(400)(0.3\theta)^2 = \text{const.}$$

$$= 0.03844\dot{\theta}^2 + 18\theta^2$$

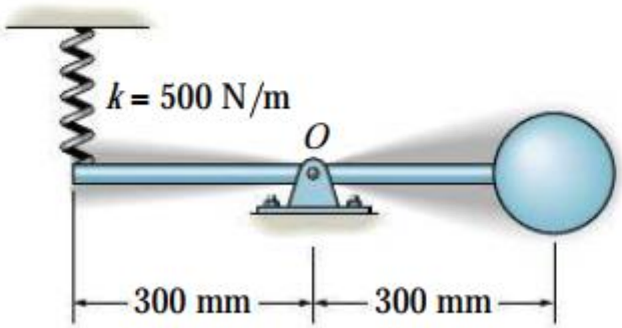
$$0.076875\dot{\theta}\ddot{\theta} + 36\theta\dot{\theta} = 0$$

$$0.076875\dot{\theta}(\ddot{\theta} + 468.29\theta) = 0 \text{ Since } 0.076875\theta \neq 0$$

$$\ddot{\theta} + 468\theta = 0$$ **Ans.**

Problem 18

22–35. Determine the natural period of vibration of the 3-kg sphere. Neglect the mass of the rod and the size of the sphere.



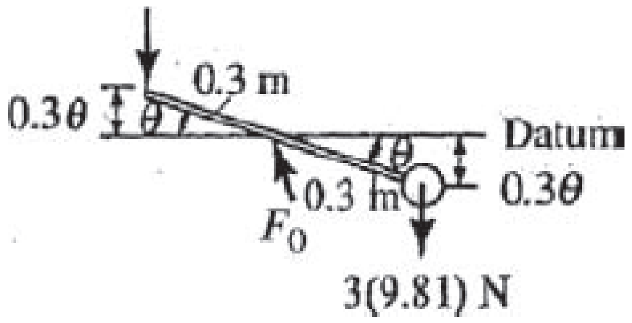
Prob. 22–35

SOLUTION

$$E = T + V$$

$$= \frac{1}{2}(3)(0.3\dot{\theta})^2 + \frac{1}{2}(500)(\delta_{st} + 0.3\theta)^2 - 3(9.81)(0.3\theta)$$

$$\dot{E} = \dot{\theta}[(3(0.3)^2\ddot{\theta} + 500(\delta_{st} + 0.3\theta)(0.3) - 3(9.81)(0.3)] = 0$$



By statics,

$$T(0.3) = 3(9.81)(0.3)$$

$$T = 3(9.81)\ \mathrm{N}$$

$$\delta_{st} = \frac{3(9.81)}{500}$$

Thus,

$$3(0.3)^2\ddot{\theta} + 500(0.3)^2\theta = 0$$

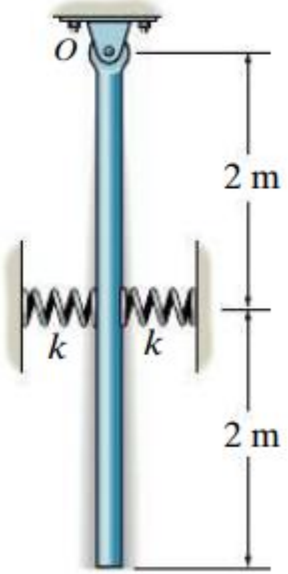
$$\ddot{\theta} + 166.67\theta = 0$$

$$\omega_n = \sqrt{166.67} = 12.91\ \mathrm{rad/s}$$

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{12.91} = 0.487\ \mathrm{s}$$ **Ans.**

Problem 19

*22–36. If the lower end of the 6-kg slender rod is displaced a small amount and released from rest, determine the natural frequency of vibration. Each spring has a stiffness of $k = 200\ \text{N/m}$ and is unstretched when the rod is hanging vertically.



Prob. 22–36

SOLUTION

Energy Equation. The mass moment of inertia of the rod about O is $I_0 = \frac{1}{3}ml^2 = \frac{1}{3}(6)(4^2) = 32\ \text{kg}\cdot\text{m}^2$. Thus, the Kinetic energy is

$$T = \frac{1}{2}I_0\omega^2 = \frac{1}{2}(32)\dot{\theta}^2 = 16\dot{\theta}^2$$

with reference to the datum set in Fig. *a*, the gravitational potential energy is

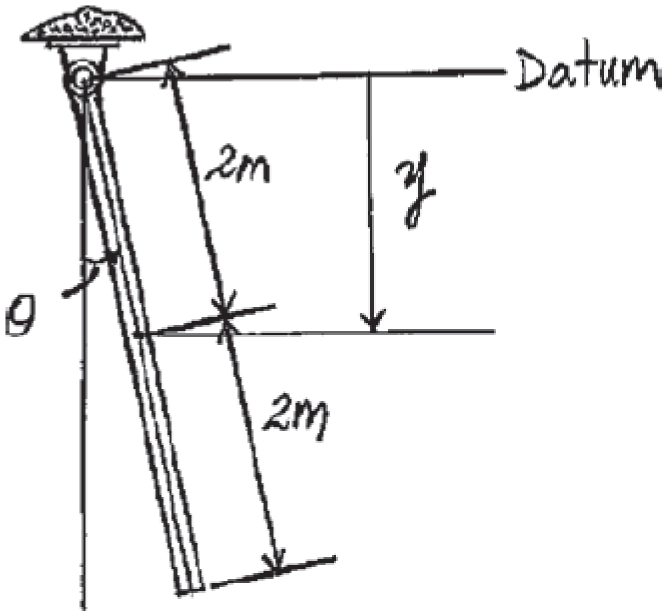
$$V_g = mgy = 6(9.81)(-2\cos\theta) = -117.72\cos\theta$$

When the rod undergoes a small angular displacement θ the spring deform $x = 2\sin\Omega$. Thus the elastic potential energy is

$$V_e = 2\left(\frac{1}{2}kx^2\right) = 2\left[\frac{1}{2}(200)(2\sin\theta)^2\right] = 800\sin^2\theta$$

Thus, the total energy is

$$E = T + V = 16\dot{\theta}^2 + 800\sin^2\theta - 117.72\cos\theta$$



(a)

Time Derivative. Taking the first time derivative of the above equation

$$32\dot{\theta}\ddot{\theta} + 1600(\sin\theta\cos\theta)\dot{\theta} + 117.72(\sin\theta)\dot{\theta} = 0$$

Using the trigonometry identity $\sin 2\theta = 2\sin\theta\cos\theta$, we obtain

$$32\dot{\theta}\ddot{\theta} + 800(\sin 2\theta)\dot{\theta} + 117.72(\sin\theta)\dot{\theta} = 0$$

$$\dot{\theta}(32\ddot{\theta} + 800\sin 2\theta + 117.72\sin\theta) = 0$$

Since $\dot{\theta} \neq 0$,

$$32\ddot{\theta} + 800\sin 2\theta + 117.72\sin\theta) = 0$$

Since θ is small, $\sin 2\theta \simeq 2\theta$ and $\sin\theta = \theta$. The above equation becomes

$$32\ddot{\theta} + 1717.72\theta = 0$$

$$\ddot{\theta} + 53.67875\theta = 0$$

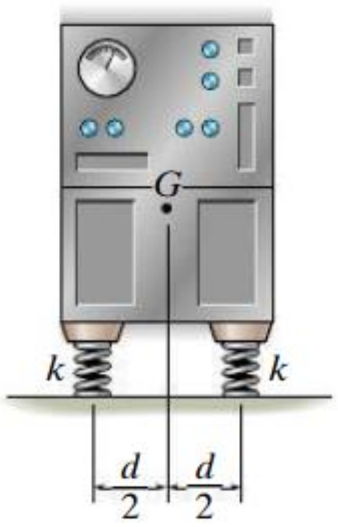
Comparing to that of standard form, $\omega_n = \sqrt{53.67875} = 7.3266\text{ rad/s}$.

Thus,

$$f = \frac{\omega_n}{2\pi} = \frac{7.3266}{2\pi} = 1.1661\text{ Hz} = 1.17\text{ Hz} \qquad \textbf{Ans.}$$

Problem 20

22–38. The machine has a mass m and is uniformly supported by *four* springs, each having a stiffness k. Determine the natural period of vertical vibration.



Prob. 22–38

SOLUTION

$$T + V = \text{const.}$$

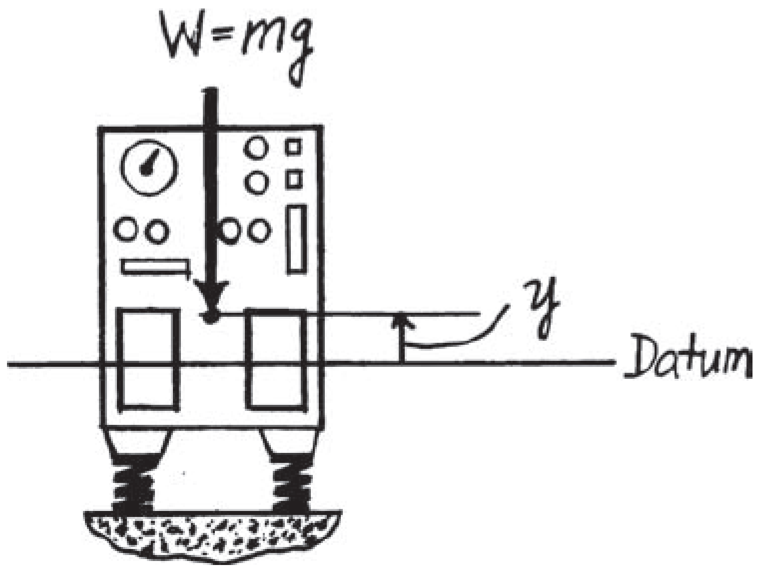
$$T = \frac{1}{2} m(\dot{y})^2$$

$$V = m\,g\,y + \frac{1}{2}(4k)(\Delta s - y)^2$$

$$T + V = \frac{1}{2} m(\dot{y})^2 + m\,g\,y + \frac{1}{2}(4k)(\Delta s - y)^2$$

$$m\,\dot{y}\,\ddot{y} + m\,g\,\dot{y} - 4k(\Delta s - y)\dot{y} = 0$$

$$m\,\ddot{y} + m\,g + 4ky - 4k\Delta s = 0$$



Since $\Delta s = \dfrac{mg}{4k}$

Then

$$m\ddot{y} + 4ky = 0$$

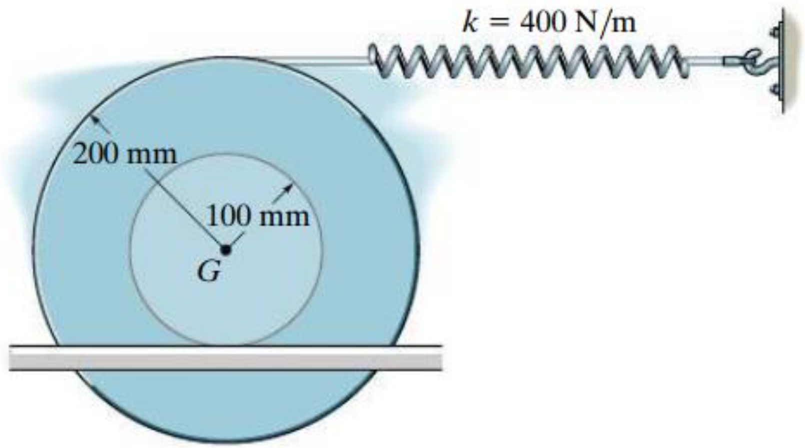
$$y + \frac{4k}{m} y = 0$$

$$\omega_n = \sqrt{\frac{4k}{m}}$$

$$\tau = \frac{2\pi}{\omega_n} = \pi\sqrt{\frac{m}{k}}$$ **Ans.**

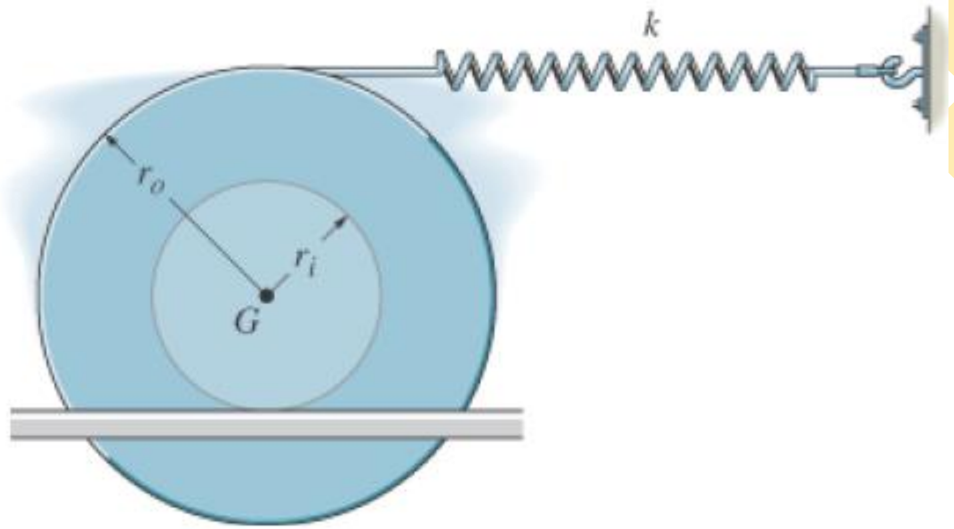
Problem 21

22–39. Determine the differential equation of motion of the 3-kg spool. Assume that it does not slip at the surface of contact as it oscillates. The radius of gyration of the spool about its center of mass is $k_G = 125$ mm.



Prob. 22–39

SOLUTION



$M = 3 \text{ kg}$

$k_G = 125 \text{ mm}$

$r_i = 100 \text{ mm}$

$r_o = 200 \text{ mm}$

$k = 400 \text{ N/m}$

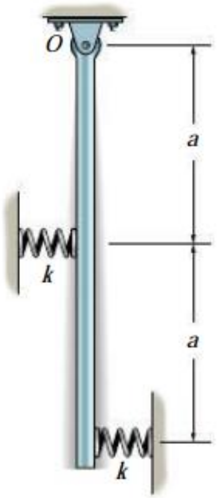
$$T + V = \frac{1}{2}M\left(k_G{}^2 + r_i{}^2\right)\theta^2 + \frac{1}{2}k\left[\left(r_o + r_i\right)\theta\right]^2$$

$$M\left(k_G{}^2 + r_i{}^2\right)\theta'' + k\left(r_o + r_i\right)^2\theta = 0 \qquad \omega_n = \sqrt{\frac{k\left(r_o + r_i\right)^2}{M\left(k_G{}^2 + r_i{}^2\right)}}$$

$$\theta'' + \omega_n{}^2\theta = 0 \qquad \text{where} \qquad \omega_n{}^2 = 468 \text{ rad}/\text{s}^2 \qquad \textbf{Ans.}$$

Problem 22

*22–40. The slender rod has a mass m and is pinned at its end O. When it is vertical, the springs are unstretched. Determine the natural period of vibration.



Prob. 22–40

SOLUTION

$$T + V = \frac{1}{2}\left[\frac{1}{3}m(2a)^2\right]\dot{\theta}^2 + \frac{1}{2}k(2\theta a)^2 + \frac{1}{2}k(\theta a)^2 + mga(1 - \cos\theta)$$

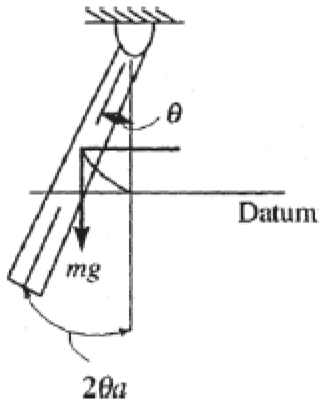
$$0 = \frac{4}{3}ma^2\dot{\theta}\,\ddot{\theta} + 4ka^2\theta\dot{\theta} + ka^2\theta\dot{\theta} + mga\sin\theta\dot{\theta}$$

$$\sin\theta = \theta$$

$$\frac{4}{3}ma^2\ddot{\theta} + 5ka^2\theta + mga\theta = 0$$

$$\ddot{\theta} + \left(\frac{15ka + 3mg}{4ma}\right)\theta = 0$$

$$\tau = \frac{2\pi}{\omega_n} = \frac{4\pi}{\sqrt{3}}\left(\frac{ma}{5ka + mg^2}\right)^{\frac{1}{2}}$$ **Ans.**



End of the Lecture

Let Learning Continue