CE3031
Mechanical Vibrations

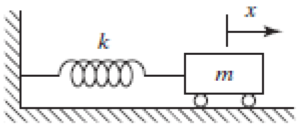
Chapter 1

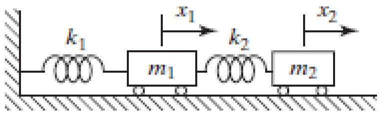
Fundamentals of Vibration

1.1 Basic concepts of vibration

- Any motion that repeats itself after an interval of time is called *vibration* or *oscillation*. The swinging of a pendulum and the motion of a plucked string are typical examples of vibration. The theory of vibration deals with the study of oscillatory motions of bodies and the forces associated with them.
- A vibratory system, in general, includes a means for storing potential energy (*spring* or elasticity), a means for storing kinetic energy (*mass* or inertia), and a means by which energy is gradually lost (*damper*).
- The mechanism by which the vibrational energy is gradually converted into heat or sound is known as *damping*. A damper is assumed to have neither mass nor elasticity, and damping force exists only if there is the relative velocity between the two ends of the damper.
- The vibration of a system involves the transfer of its potential energy to kinetic energy and of kinetic energy to potential energy, alternately. If the system is damped, some energy is dissipated in each cycle of vibration and must be replaced by an external source if a state of steady vibration is to be maintained.

- The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time defines *the number of degrees of freedom* of the system.
- Many practical systems can be described using a finite number of degrees of freedom. Some systems, especially those involving continuous elastic members, have an infinite number of degrees of freedom.
- Systems with a finite number of degrees of freedom are called *discrete* or *lumped parameter* systems, and those with an infinite number of degrees of freedom are called *continuous* or *distributed* systems.


k
m
x


k1
m1
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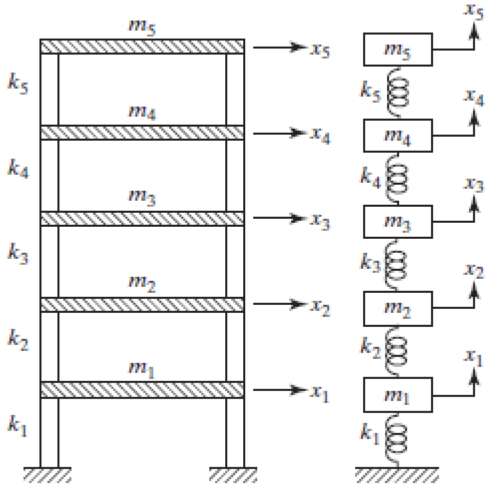
m5
m4
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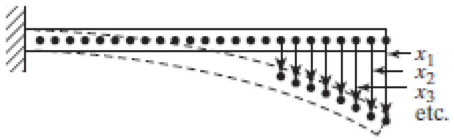
(a) (b)

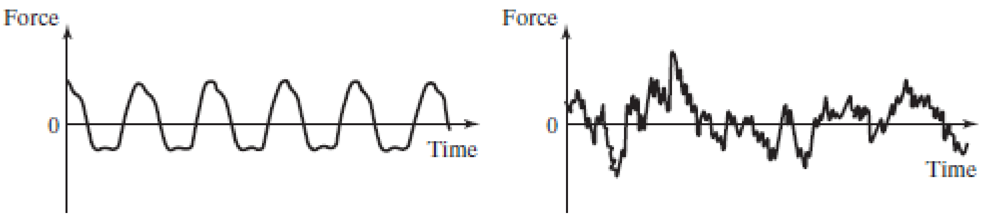

x1
x2
x3
etc.

1.2 Classification of vibration

- Vibration can be classified in several ways. Some of the important classifications are as follows.
- *Free Vibration.* If a system, after an initial disturbance, is left to vibrate on its own, the ensuing vibration is known as free vibration. No external force acts on the system. The oscillation of a simple pendulum is an example of free vibration.
- *Forced Vibration.* If a system is subjected to an external force (often, a repeating type of force), the resulting vibration is known as forced vibration. The oscillation that arises in machines such as diesel engines is an example of forced vibration.
- If the frequency of the external force coincides with one of the natural frequencies of the system, a condition known as *resonance* occurs, and the system undergoes dangerously large oscillations. Failures of such structures as buildings, bridges, turbines, and airplane wings have been associated with the occurrence of resonance.

- If no energy is lost or dissipated in friction or other resistance during oscillation, the vibration is known as *undamped vibration*. If any energy is lost in this way, however, it is called *damped vibration*. In many physical systems, the amount of damping is so small that it can be disregarded for most engineering purposes. However, consideration of damping becomes extremely important in analyzing vibratory systems near resonance.
- If all the basic components of a vibratory system—the spring, the mass, and the damper— behave linearly, the resulting vibration is known as *linear vibration*. If, however, any of the basic components behave nonlinearly, the vibration is called *nonlinear vibration*.
- If the value or magnitude of the excitation (force or motion) acting on a vibratory system is known at any given time, the excitation is called deterministic. The resulting vibration is known as *deterministic vibration*.

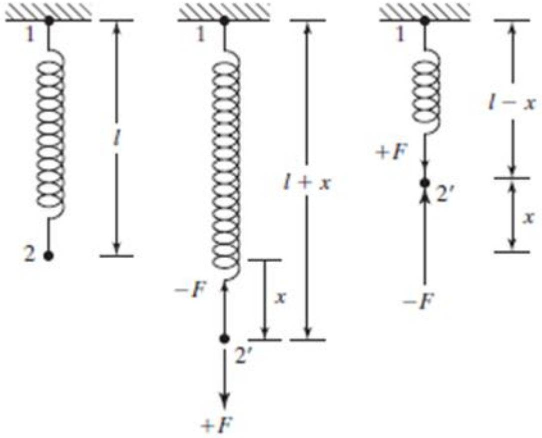
- In some cases, the excitation is *nondeterministic* or *random*, the value of the excitation at a given time cannot be predicted. In these cases, a large collection of records of the excitation may exhibit some statistical regularity. It is possible to estimate averages such as the mean and mean square values of the excitation. Examples of random excitations are wind velocity, road roughness, and ground motion during earthquakes.
- If the excitation is random, the resulting vibration is called *random vibration*. In this case, the vibratory response of the system is also random; it can be described only in terms of statistical quantities.



(a) A deterministic (periodic) excitation (b) A random excitation

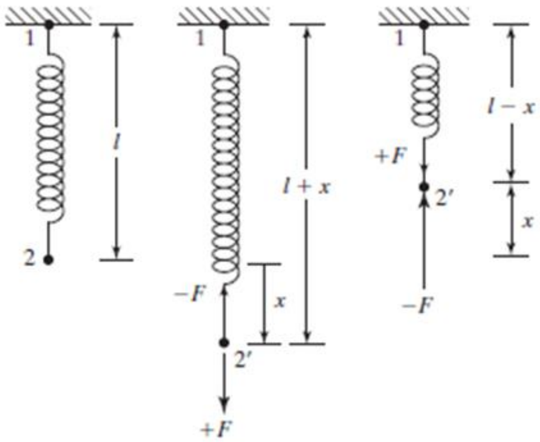
1.3 Spring elements

- Spring is a type of mechanical link, which in most applications is assumed to have negligible mass and damping. The most common type of spring is the helical-coil spring used in retractable pens and pencils, staplers, and suspensions of freight trucks and other vehicles.
- Several other types of springs can be identified in engineering applications. In fact, any elastic or deformable body or members, such as a cable, bar, beam, shaft, or plate, can be considered as a spring. Spring is commonly represented as shown.



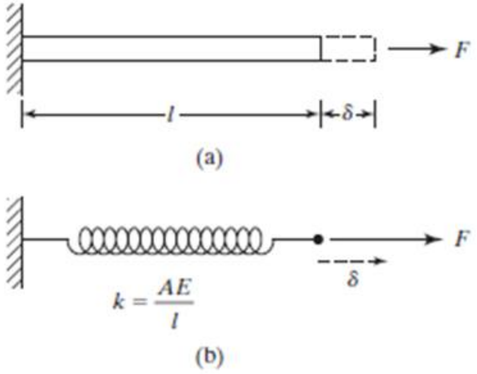
- A spring is said to be linear if the elongation or reduction in length x is related to the applied force F as

$$F = kx$$



- where k is a constant, known as the spring constant or spring stiffness, or spring rate. The spring constant k is always positive and denotes the force (positive or negative) required to cause a unit deflection (elongation or reduction in length) in the spring.

Example 1. Spring constant of a rod.



- For an axially loaded bar with cross-sectional area A, we know that

$$\delta = \frac{Fl}{EA}$$

- For the equivalent spring, we have

$$F = k_{eq}x$$

- Arranging the 1st equation in the form of the 2nd one, we have

$$F = \frac{EA}{l}\delta$$

- Combining the last two expressions, we reached the result

$$k_{eq} = \frac{EA}{l}$$

Example 2. Spring constant of a cantilever beam.

- For a cantilever beam subjected to force W at its tip, we know that

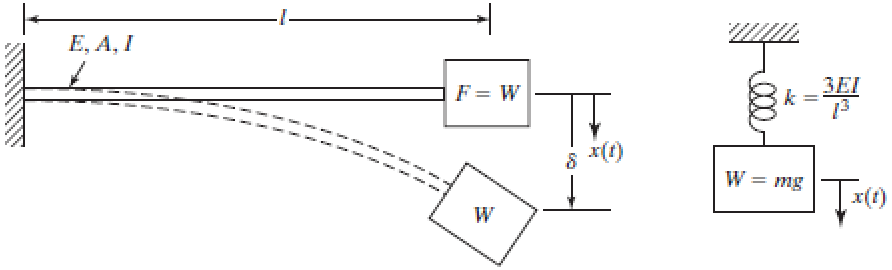
$$\delta = \frac{Wl^3}{3EI}$$

- Re-arranging the above as in the form

$$W = F = \frac{3EI}{l^3}\delta$$

- and combining $F = k_{eq}x$, we have

$$k_{eq} = \frac{3EI}{l^3}$$



(a) Cantilever with end force (b) Equivalent spring

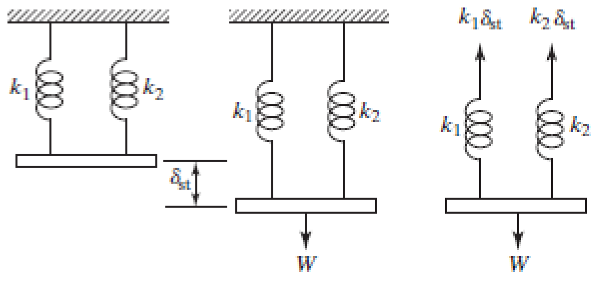
Combination of springs

- In many practical applications, several linear springs are used in combination. These springs can be combined into a single equivalent spring as indicated below.
- Case 1: Springs in Parallel
- To derive an expression for the equivalent spring constant of springs connected in parallel, consider the two springs shown. When a load W is applied, the system undergoes a static deflection δ_{st} as shown. Then the free-body diagram gives the equilibrium equation

$$W = k_1\delta_{st} + k_2\delta_{st} = (k_1 + k_2)\delta_{st} = k_{eq}\delta_{st}$$

- In general, if we have n springs with spring constants k_1, k_2,…, k_n in parallel, then the equivalent spring constant k_{eq} can be obtained:

$$k_{eq} = k_1 + k_2 + \cdots + k_n = \sum k_i$$



(a) (b) (c)

- <u>Case 2: Springs in Series</u>
- Next, we derive an expression for the equivalent spring constant of springs connected in series by considering the two springs shown. Under the action of a load W, springs 1 and 2 undergo elongations δ_1 and δ_2, respectively, as shown. The total elongation (or static deflection) of the system, δ_{st}, is given by

$$\delta_{st} = \delta_1 + \delta_2$$

- Since both springs are subjected to the same force W, we have the equilibrium shown

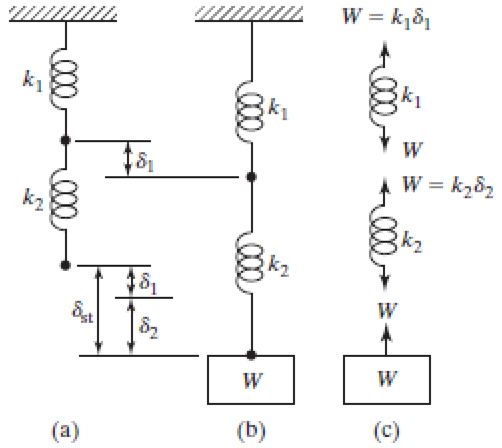
$$W = k_1\delta_1, \qquad W = k_2\delta_2$$

- If k_{eq} denotes the equivalent spring constant, then for the same static deflection

$$W = k_{eq}\delta_{st}$$

- The last three equations give

$$k_1\delta_1 = k_2\delta_2 = k_{eq}\delta_{st}$$



- or

$$\delta_1 = \frac{k_{eq}\delta_{st}}{k_1}, \qquad \delta_2 = \frac{k_{eq}\delta_{st}}{k_2}$$

- Substituting the δ_1 and δ_2 into $\delta_{st} = \delta_1 + \delta_2$, we obtain

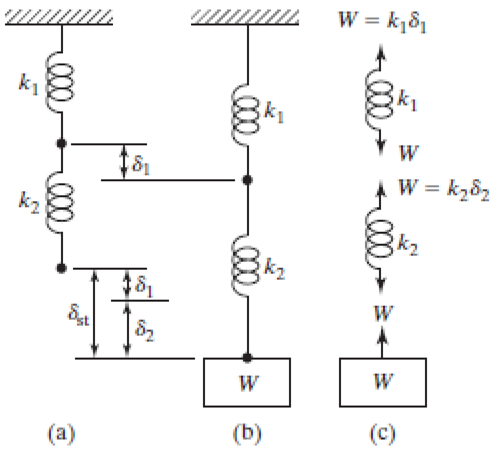
$$\frac{k_{eq}\delta_{st}}{k_1} + \frac{k_{eq}\delta_{st}}{k_2} = \delta_{st}$$

- from which

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

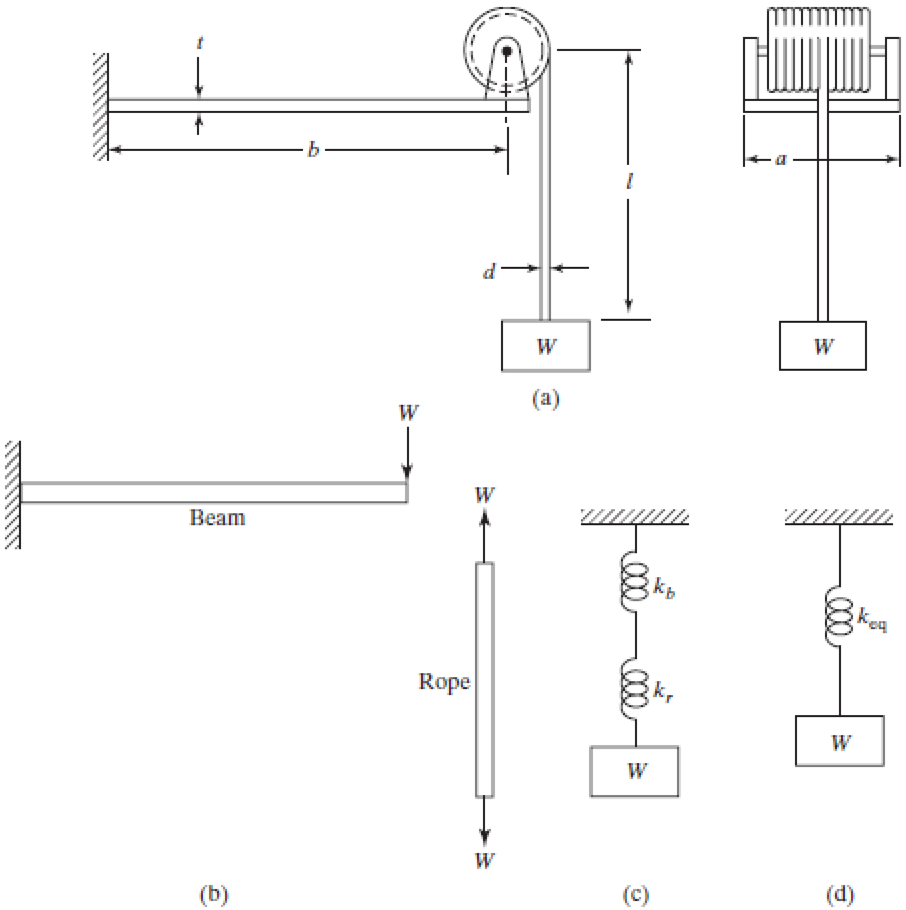
- In general, if we have *n* springs with spring constants k_1, k_2,…, k_n in series, then the equivalent spring constant k_{eq} can be obtained:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} = \sum \frac{1}{k_i}$$



Example 3.

- A hoisting drum, carrying a steel wire rope, is mounted at the end of a cantilever beam as shown. Determine the equivalent spring constant of the system when the suspended length of the wire rope is l.
- Assume that the net cross-sectional diameter of the wire rope is *d* and Young's modulus of the beam and the wire rope is *E*.



- The spring constant of the cantilever beam is given by

$$k_{beam} = \frac{3EI}{b^3} = \frac{3E}{b^3}\frac{at^3}{12} = \frac{Eat^3}{4b^3}$$

- The stiffness of the wire rope subjected to axial loading is

$$k_{rope} = \frac{EA}{l} = \frac{E\pi d^2}{4l}$$

- Since both the wire rope and the cantilever beam experience the same load *W* as shown, they can be modeled as springs in series. The equivalent spring constant k_{eq} is given by

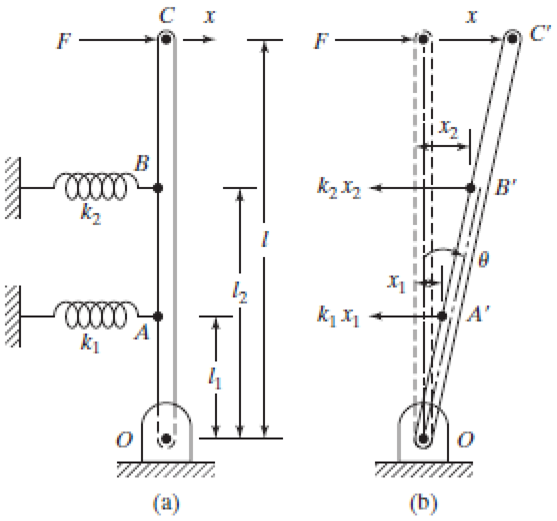
$$\frac{1}{k_{eq}} = \frac{1}{k_{beam}} + \frac{1}{k_{rope}} = \frac{4b^3}{Eat^3} + \frac{4l}{E\pi d^2}$$

- Thus,

$$k_{eq} = \frac{E}{4}\left(\frac{\pi a t^3 d^2}{\pi d^2 b^3 + l a t^3}\right)$$

Example 4.

- A hinged rigid bar of length l is connected by two springs of stiffnesses k_1 and k_2 and is subjected to a force F as shown. Assuming the angular displacement of the bar θ is small, find the equivalent spring constant of the system that relates the applied force F to the resulting displacement x.



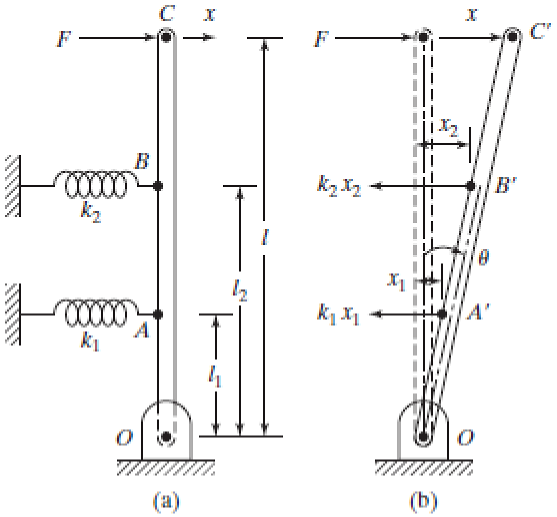
- The equivalent spring constant of the system k_{eq} referred to the point of application of the force F can be determined by considering the moment equilibrium of the forces about the hinge point O:

$$k_1 x_1 l_1 + k_2 x_2 l_2 = Fl$$

$$F = \frac{k_1 x_1 l_1}{l} + \frac{k_2 x_2 l_2}{l} = k_{eq} x$$

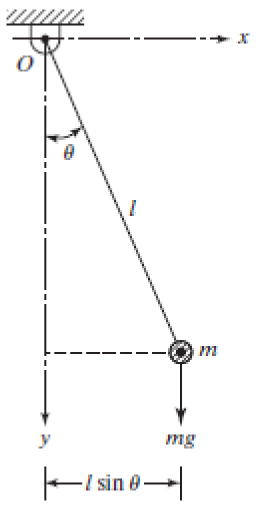
- Using $x_1 = l_1\theta$, $x_2 = l_2\theta$ and $x = l\theta$

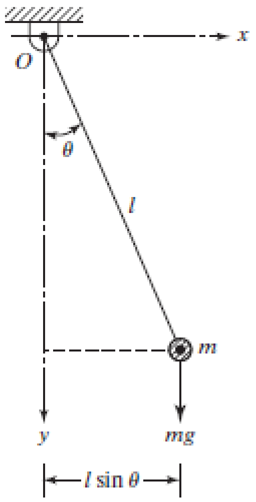
$$k_{eq} = k_1\left(\frac{l_1}{l}\right)^2 + k_2\left(\frac{l_2}{l}\right)^2$$



Example 5.

- Figure shows a simple pendulum of length l with a bob of mass m. Considering an angular displacement θ of the pendulum, determine the equivalent spring constant associated with the restoring force (or moment).





- When the pendulum undergoes an angular displacement θ, the mass m moves by a distance $l \sin\theta$ along the horizontal (*x*) direction. The restoring moment (*T*) created by the weight of the mass (*mg*) about the pivot point O is given by

$$T = mgl\sin\theta$$

- For small angular displacements θ, note that $\sin\theta \approx \theta$, thus

$$T = mgl\theta$$

- Since $T = k_{eq}\theta$

$$k_{eq} = mgl$$

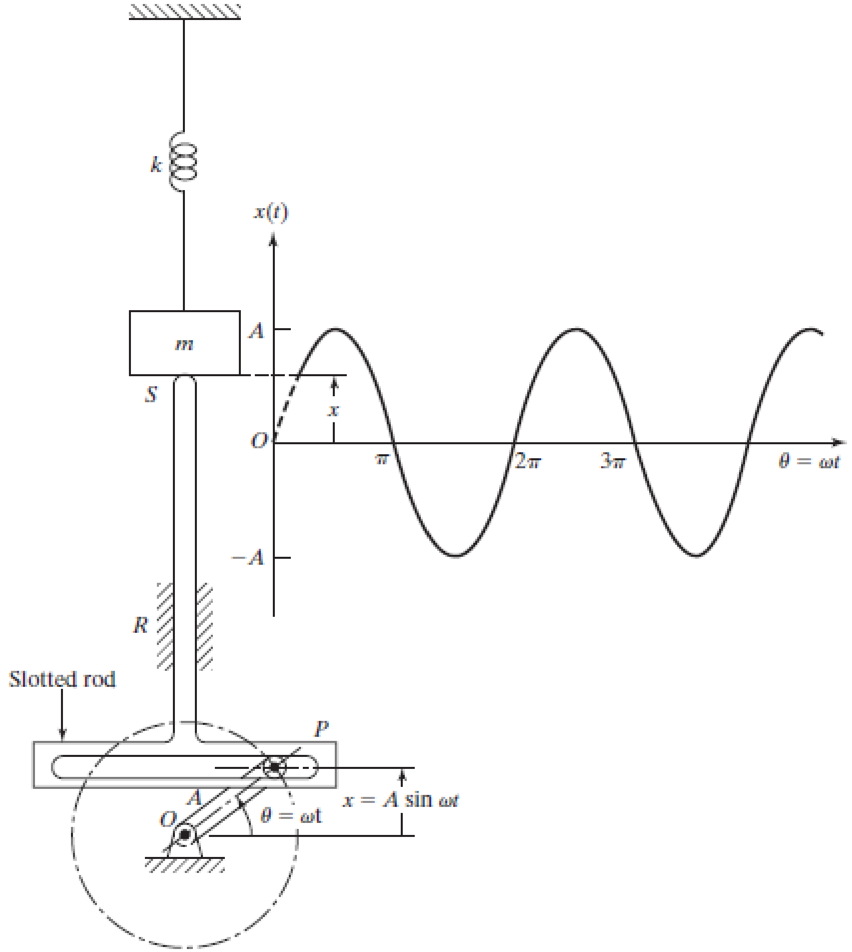
1.4 Harmonic motion

- Oscillatory motion may repeat itself regularly, as in the case of a simple pendulum, or it may display considerable irregularity, as in the case of ground motion during an earthquake. If the motion is repeated after equal intervals of time, it is called *periodic motion*. The simplest type of periodic motion is *harmonic motion*.

- The motion of the mass or its displacement is defined by

$$x = A \sin \omega t$$

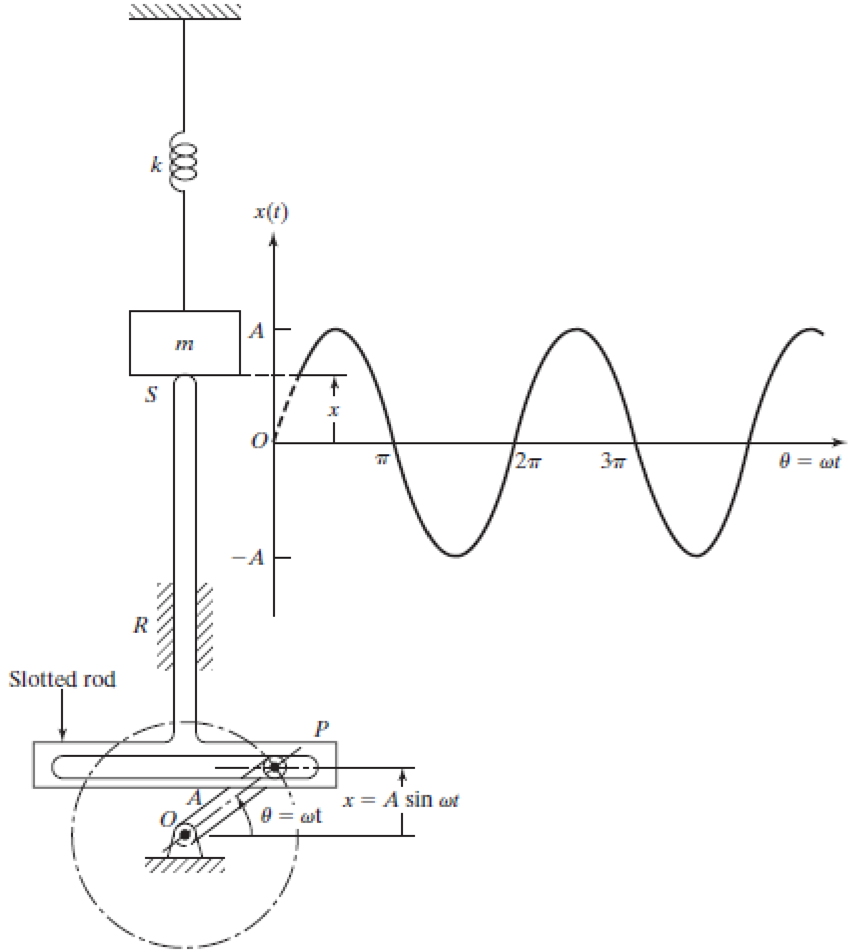
- where $Athe$ is amplitude of the vibration, and ω is the forcing frequency (angular velocity).



1.5 Definitions and terminology

- **Cycle.** The movement of a vibrating body from its undisturbed o equilibrium position to its extreme position in one direction, then t the equilibrium position, then to its extreme position in the othe direction, and back to the equilibrium position is called a *cycle o vibration*.

- **Amplitude.** The maximum displacement of a vibrating body from its equilibrium position is called the *amplitude of vibration*.



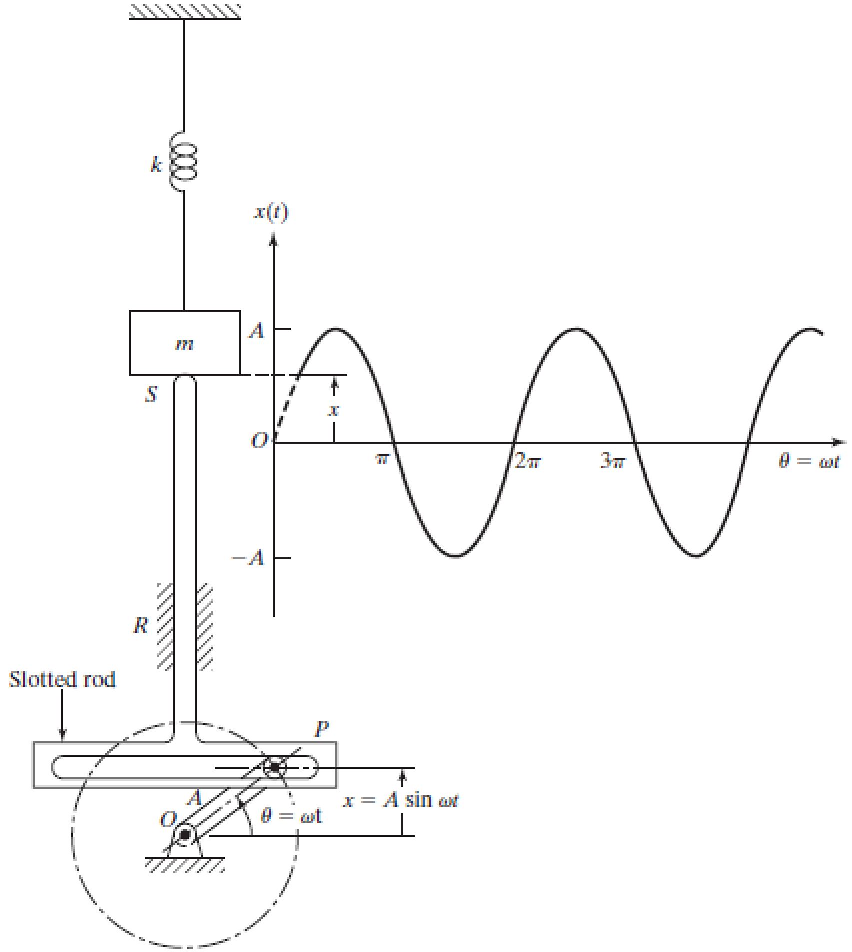
- **Period of oscillation.** The time taken to complete one cycle of motion is known as the *period of oscillation* or *period* and is denoted by τ in seconds.

$$\tau = \frac{2\pi}{\omega}$$

- where ω is called the circular frequency in rad/s.

- **Frequency of oscillation.** The number of cycles per unit time is called the frequency of oscillation or simply the frequency and is denoted by f in Hertz (Hz).

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi}$$



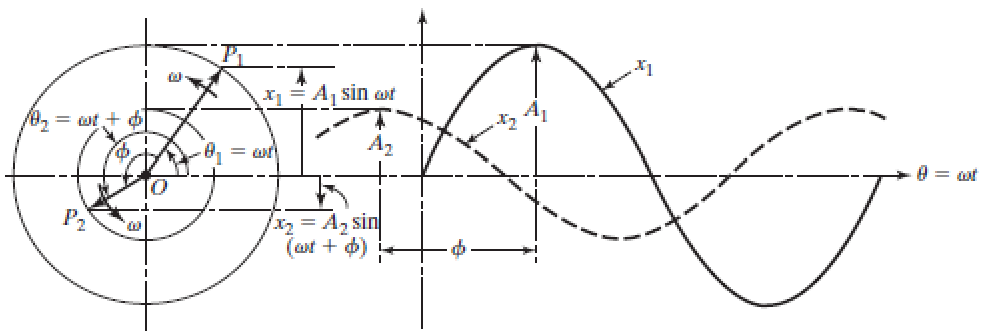
- **Phase angle.** Consider two vibratory motions denoted by

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin(\omega t + \phi)$$

- The two harmonic motions given by the above equations are called synchronous because they have the same frequency or angular velocity ω. Two synchronous oscillations need not have the same amplitude, and they need not attain their maximum values at the same time.

- Based on the graphical representation of these two motions, the second vector OP_2 leads the first one OP_1 by an angle ϕ, known as the *phase angle*. This means that the maximum of the second vector would occur ϕ radians earlier than that of the first vector.



- **Natural frequency.** If a system, after an initial disturbance, is left to vibrate on its own, the frequency with which it oscillates without external forces is known as its natural frequency. A vibratory system having *n* degrees of freedom will have, in general, *n* distinct natural frequencies of vibration.

- **Beats.** When two harmonic motions, with frequencies close to one another, are added, the resulting motion exhibits a phenomenon known as *beats*.

- **Octave.** When the maximum value of a range of frequency is twice its minimum value, it is known as an octave band. For example, each of the ranges 75–150 Hz, 150–300 Hz, and 300–600 Hz can be called an octave band. In each case, the maximum and minimum values of frequency, which have a ratio of 2:1, are said to differ by an *octave*.

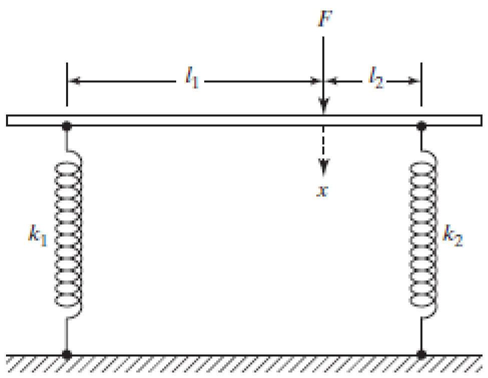
- **Decibel.** The various quantities encountered in the field of vibration and sound (such as displacement, velocity, acceleration, pressure, and power) are often represented using the notation of *decibels*.

$$dB = 10 \log \frac{P}{P_0} = 20 \log \frac{X}{X_0}$$

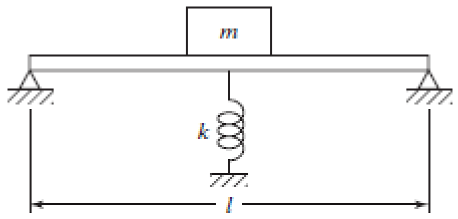
- where P_0 is some reference value of power, and X_0 is a specified reference voltage.

Review Problems

- **Problem 1.1** Consider a system of two springs, with stiffnesses k_1 and k_2, arranged in parallel as shown. The rigid bar to which the two springs are connected remains horizontal when the force F is zero. Determine the equivalent spring constant of the system k_e that relates the force applied (F) to the resulting displacement (x).



- **Problem 1.2** A machine of mass m = 500 kg is mounted on a simply supported steel beam of length l = 2 m having a rectangular cross-section (depth = 0.1 m, width = 1.2 m) and Young's modulus E = 206 GPa. To reduce the vertical deflection of the beam, a spring of stiffness k is attached at the midspan as shown. Determine the value of k needed to reduce the deflection of the beam by



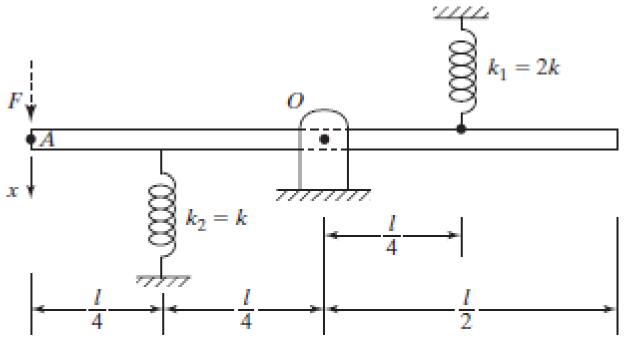
a) 25 percent of its original value.

b) 50 percent of its original value.

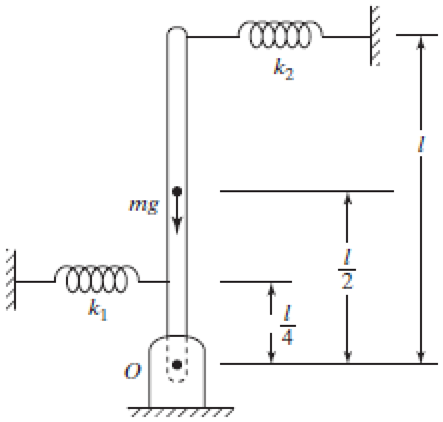
c) 75 percent of its original value.

- Assume that the mass of the beam is negligible.

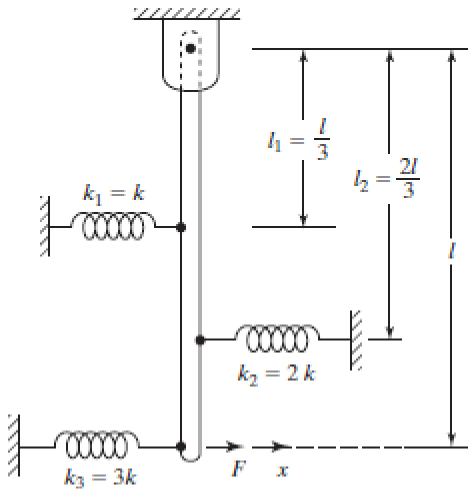
- **Problem 1.3** The static equilibrium position of a massless rigid bar, hinged at point O and connected with springs k_1 and k_2, is shown. Assuming the displacement (x) resulting from the application of a force F at point A is small, find the equivalent spring constant of the system, k_e, that relates the applied force F to the displacement x.



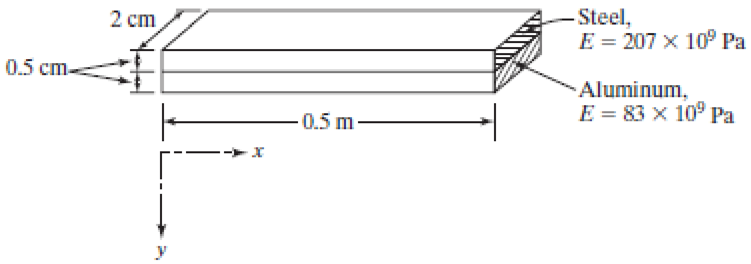
- **Problem 1.4** Figure shows a uniform rigid bar of mass m that is pivoted at point O and connected by springs of stiffnesses k_1 and k_2. Considering a small angular displacement θ of the rigid bar about the point O, determine the equivalent spring constant associated with the restoring moment.



- **Problem 1.5** Derive the expression for the equivalent spring constant that relates the applied force *F* to the resulting displacement *x* of the system shown. Assume the displacement of the link to be small.



- **Problem 1.6** Find the spring constant of the bimetallic bar shown in axial motion.



End of the Lecture

Let Learning Continue